Unit step response of second-order systems

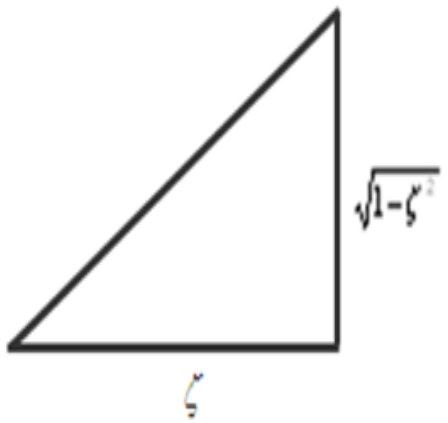
Suppose, $r(t) = u(t), \Rightarrow R(s) = \frac{1}{s}; \quad Y(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Or,

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} = \frac{1}{s} - \frac{s + \zeta\omega_n + \zeta\omega_n}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2}$$

Performing inverse Laplace transform,

$$y(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1 - \zeta^2})t - e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1 - \zeta^2})t \cdot \frac{\zeta}{\sqrt{1 - \zeta^2}}$$



$$\text{or, } y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2})t + \zeta\sin(\omega_n\sqrt{1-\zeta^2})t\right]$$

$$\text{or, } y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta), \text{where, } \omega_d = \omega_n\sqrt{1-\zeta^2} \text{ and } \theta = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right) = \cos^{-1}\zeta$$

$$\text{or, } y(t) = 1 - \frac{\omega_n}{\omega_d}e^{-\sigma t}\sin(\omega_d t + \theta)$$

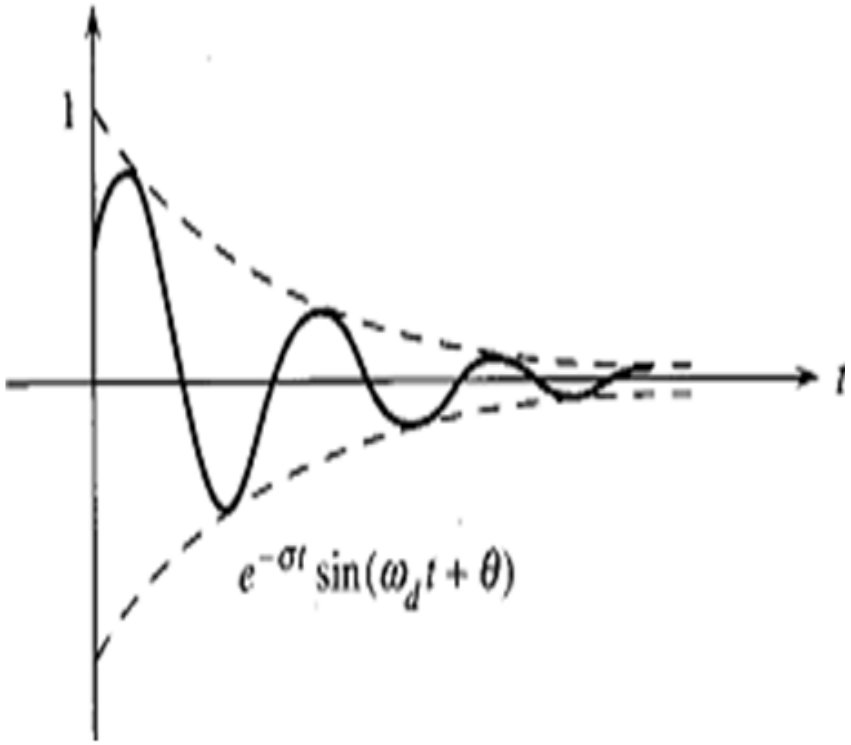
The plot of $e^{-\sigma t}\sin(\omega_d t+\theta)$ is shown in Figure.

The steady-state response is,

$$y_{ss}(t)=\lim_{t\to\infty} y(t)=1$$

Thus, the system has zero steady-state error.
The pole of $T(s)$ dictates the response,

$$e^{-\sigma t}\sin(\omega_d t+\theta).$$



Time response specifications

Control systems are generally designed with damping less than one, i.e., oscillatory step response. Higher order control systems usually have a pair of complex conjugate poles with damping less than unity that dominate over the other poles. Therefore the time response of second- and higher-order control systems to a step input is generally of damped oscillatory nature as shown in Figure next (next page).

In specifying the transient-response characteristics of a control system to a unit step input, we usually specify the following:

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Peak overshoot, M_p
5. Settling time, t_s
6. Steady-state error, e_{ss}