Continuous-time (CT) system analysis using LT

CT, LTI Systems

- Consider the following CT LTI system:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Assumption: the impulse response *h*(*t*) is ***absolutely integrable,*** i.e.,

$$\int |h(t)|\, dt < \infty$$

(this has to do with system stability)

Response of a CT, LTI System to a Sinusoidal Input

- What's the response $y(t)$ of this system to the input signal

$$x(t) = \cos(\omega_0 t + \theta), \;\; t \in \mathbb{R} \qquad ?$$

- We start by looking for the response $y_c(t)$ of the same system to

$$x_c(t) = e^{j\omega_0 t} \qquad t \in \mathbb{R}$$

Response of a CT, LTI System to a Complex Exponential Input

- The output is obtained through convolution as

$$y_c(t) = h(t) * x_c(t) = \int_{\mathbb{R}} h(\tau) x_c(t-\tau) d\tau =$$

$$= \int_{\mathbb{R}} h(\tau) e^{j\omega_0 (t-\tau)} d\tau =$$

$$= \underbrace{e^{j\omega_0 t}}_{x_c(t)} \int_{\mathbb{R}} h(\tau) e^{-j\omega_0 \tau} d\tau =$$

$$= x_c(t) \int_{\mathbb{R}} h(\tau) e^{-j\omega_0 \tau} d\tau$$

The Frequency Response of a CT, LTI System

- By defining

$$H(\omega) = \int_{\mathbb{R}} h(\tau) e^{-j\omega\tau} d\tau$$

$H(\omega)$ is the frequency response of the CT, LTI system = Fourier transform of *h*(*t*)

it is

$$y_c(t) = H(\omega_0) x_c(t) =$$
$$= H(\omega_0) e^{j\omega_0 t}, \quad t \in \mathbb{R}$$

- Therefore, the response of the LTI system to a complex exponential is another complex exponential with the same frequency ω_0

Analyzing the Output Signal $y_c(t)$

- Since $H(\omega_0)$ is in general a complex quantity, we can write

$$y_c(t) = H(\omega_0)e^{j\omega_0 t} =$$

$$= |H(\omega_0)|\, e^{j\arg H(\omega_0)} e^{j\omega_0 t} =$$

$$= \underbrace{|H(\omega_0)|}_{\text{output signal's magnitude}} e^{j(\omega_0 t + \underbrace{\arg H(\omega_0)}_{\text{output signal's phase}})}$$

Response of a CT, LTI System to a Sinusoidal Input

- With Euler's formulas we can express x(t) as

$$x(t) = \cos(\omega_0 t + \theta)$$

$$= \tfrac{1}{2}(e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)})$$

$$= \tfrac{1}{2}e^{j\theta}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

Using the previous result, the response is

$$y(t) = \tfrac{1}{2}e^{j\theta}H(\omega_0)e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}H(-\omega_0)e^{-j\omega_0 t}$$

Response of a CT, LTI System to a Sinusoidal Input – Cont'd

- If h(t) is real, then $H(-\omega) = H^*(\omega)$ and

$$H(\omega_0) = |H(\omega_0)|\, e^{j \arg H(\omega_0)}$$

$$H(-\omega_0) = |H(\omega_0)|\, e^{-j \arg H(\omega_0)}$$

- Thus we can write y(t) as

$$y(t) = \frac{1}{2}|H(\omega_0)|\, e^{j(\omega_0 t + \theta + \arg H(\omega_0))} + \frac{1}{2}|H(\omega_0)|\, e^{-j(\omega_0 t + \theta + \arg H(\omega_0))}$$

$$= |H(\omega_0)| \cos(\omega_0 t + \theta + \arg H(\omega_0))$$

Response of a CT, LTI System to a Sinusoidal Input – Cont'd

- Thus, the response to

$$x(t) = A\cos(\omega_0 t + \theta)$$

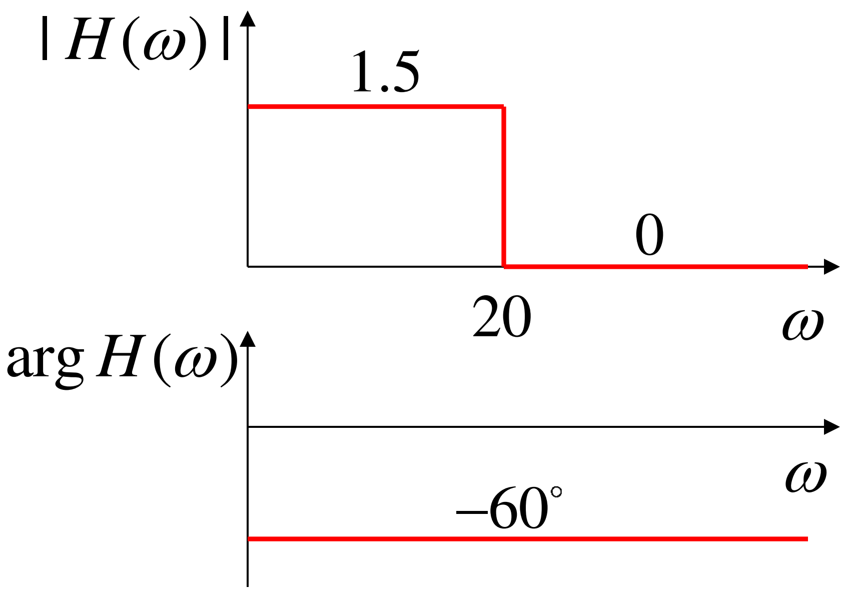
is

$$y(t) = A\,|\,H(\omega_0)\,|\cos\left(\omega_0 t + \theta + \arg H(\omega_0)\right)$$

which is also a sinusoid with the same frequency ω_0 but with the amplitude scaled by the factor $|\,H(\omega_0)\,|$ and with the phase shifted by amount $\arg H(\omega_0)$

Example: Response of a CT, LTI System to Sinusoidal Inputs

- Suppose that the frequency response of a CT, LTI system is defined by the following specs:



$$|H(\omega)| = \begin{cases} 1.5, & 0 \le \omega \le 20, \\ 0, & \omega > 20, \end{cases}$$

$$\arg H(\omega) = -60^\circ, \forall \omega$$