UNIT-2

(Lecture-2)

State Equations

State Equations

- **Let us define the state of the system by an *N*-element column vector, *x*(*t*):**

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_N(t) \end{bmatrix}^t$$

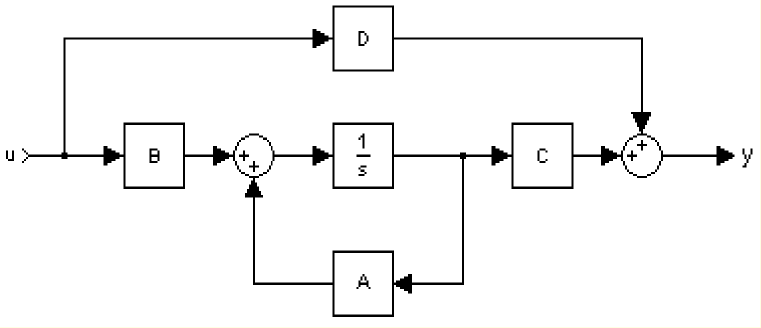
Note that in this development, *v*(*t*) will be the input, *y*(*t*) will be the output, and *x*(*t*) is used for the state variables.

- **Any system can be modeled by the following state equations:**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t) \qquad \mathbf{x} : Nx1 \quad \mathbf{A} : NxN \quad \mathbf{B} : Nxp \quad p : \text{number of inputs}$$

$$\mathbf{y}(t) = \mathbf{C}x(t) + \mathbf{D}\mathbf{v}(t) \qquad \mathbf{y} : qx1 \quad \mathbf{C} : qxN \quad \mathbf{D} : qxp \quad q : \text{number of outputs}$$

- **This system model can handle single input/single output systems, or multiple inputs and outputs.**
- **The equations above can be Implemented using the signal flow graph shown to the right.**
- **Works for ALL linear systems!**



Differential Equations

- **Consider the CT differential equations:**

$$\ddot{y}(t)+a_1\dot{y}(t)+a_0 y(t)=b_0 v(t)$$

- **A second-order differential equation requires two state variables:**

$$x_1(t)=y(t) \qquad x_2(t)=\dot{y}(t)$$

- **We can reformulate the differential equation as a set of three equations:**

$$\dot{x}_1(t)=x_2(t)$$
$$\dot{x}_2(t)=-a_0x_1(t)-a_1x_2(t)+b_0v(t)$$
$$y(t)=x_1(t)$$

- **We can write these in matrix form as:**

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ b_0\end{bmatrix}v(t)$$

$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

- **This can be extended to an *N*th-order differential equation of this type:**

$$y^{(N)}(t)+\sum_{i=0}^{N-1}a_i y^{(i)}(t)=b_0v(t)$$

- **The state variables are defined as:** $x_i(t)=y^{(i-1)}(t), \quad i=1, 2, \ldots, N$

Differential Equations (Cont.)

- **The resulting state equations are:**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\vdots$$
$$\dot{x}_{N-1}(t) = x_N(t)$$
$$\dot{x}_N(t) = -\sum_{i=0}^{N-1} a_i x_{i+1}(t) + b_0 v(t)$$
$$y(t) = x_1(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \mathbf{D} = 0$$

- **Next, consider a differential equation with a more complex forcing function:**

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_1\dot{v}(t) + b_0 v(t)$$

- **The state model is:**

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \qquad y(t) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- **We can verify this by expanding the matrix equation:**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -a_0 x_1(t) - a_1 x_2(t) + v(t)$$
$$y(t) = b_0 x_1(t) + b_1 x_2(t)$$

Differential Equations (Cont.)

- **To construct the original equation, differentiate the last equation:**

$$\begin{aligned}\dot{y}(t) &= b_0\dot{x}_1(t) + b_1\dot{x}_2(t)\\ &= b_0x_2(t) + b_1\left[-a_0x_1(t) - a_1x_2(t) + v(t)\right]\\ &= -a_1y(t) + \left(a_1b_0 - a_0b_1\right)x_1(t) + b_0x_2(t) + b_1v(t)\end{aligned}$$

- **Differentiate the last equation again and substitute:**
- **Hence, given a general LTI system:**

$$\begin{aligned}\ddot{y}(t) &= -a_1\dot{y}(t) + \left(a_1b_0 - a_0b_1\right)\dot{x}_1(t) + b_0\dot{x}_2(t) + b_1\dot{v}(t)\\ &= -a_1\dot{y}(t) + \left(a_1b_0 - a_0b_1\right)x_2(t)\\ &\quad + b_0\left[-a_0x_1(t) - a_1x_2(t) + v(t)\right] + b_1\dot{v}(t)\\ &= -a_1\dot{y}(t) - b_0a_0x_1(t) - a_0b_1x_2(t) + b_0v(t) + b_1\dot{v}(t)\\ &= -a_1\dot{y}(t) - a_0\left(b_0x_1(t) + b_1x_2(t)\right) + b_0v(t) + b_1\dot{v}(t)\\ &= -a_1\dot{y}(t) - a_0y(t) + b_0v(t) + b_1\dot{v}(t)\end{aligned}$$

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = b_0v(t) + b_1\dot{v}(t)$$

$$y^{(N)}(t) + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = \sum_{i=0}^{N-1} b_i v^{(i)}(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1}\end{bmatrix} \quad \mathbf{B} = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_N\end{bmatrix} \quad \mathbf{D} = 0$$