UNIT-2

(Lecture-8)

Solutions and Summary

Solutions to the State Equations – Preliminaries

- **Recall our state equations:**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t)$$

$$\mathbf{y}(t) = \mathbf{C}x(t) + \mathbf{D}\mathbf{v}(t)$$

- **To solve these equations, we will need a few mathematical tools. First:**

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$$

where I is an *N*x*N* identity matrix. A^k is simply AxAx…A.

- **For any real numbers *t* and λ:**

$$e^{\mathbf{A}(t+\lambda)} = e^{\mathbf{A}t} e^{\mathbf{A}\lambda}$$

- **Further, setting λ = -*t*:**

$$e^{\mathbf{A}(t+\lambda)} = e^{\mathbf{A}t} e^{-\mathbf{A}t} = \mathbf{I}$$

- **Next:**

$$\frac{d}{dt} e^{\mathbf{A}t} = \frac{d}{dt}\left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots\right] = \mathbf{A} + \mathbf{A}^2 t + \frac{\mathbf{A}^3 t^2}{2!} + \dots$$

$$= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots\right) = \mathbf{A}e^{\mathbf{A}t}$$

- **We can use these results to show that the solution to** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ **is:**

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0), \quad t \ge 0$$

Solutions to the Forced Equation

- **If:** $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0), \quad t \geq 0$

$$\frac{d}{dt}\left[\mathbf{x}(t)\right] = \frac{d}{dt}\left[e^{\mathbf{A}t}\mathbf{x}(0)\right] = \frac{d}{dt}\left[e^{\mathbf{A}t}\right]\mathbf{x}(0) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{A}\mathbf{x}(t)$$

- $e^{\mathbf{A}t}$ **is referred to as the state-transition matrix.**
- **We can apply these results to the state equations:**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t)$$

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{v}(t)$$

$$e^{-\mathbf{A}t}\left[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{v}(t)$$

- **Note that:**

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) + \left[\frac{d}{dt}e^{-\mathbf{A}t}\right]\mathbf{x}(t) = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - \mathbf{A}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\left[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right]$$

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{v}(t)$$

- **Integrating both sides:**

$$e^{-\mathbf{A}t}\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\lambda}\mathbf{B}\mathbf{v}(\lambda)\,d\lambda$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}(t-\lambda)}\mathbf{B}\mathbf{v}(\lambda)\,d\lambda, \quad t \geq 0$$

Generalization of our convolution integral

Solution to the Output Equation

- **Recall:**

$$\mathbf{y}(t)=\mathbf{C}x(t)+\mathbf{Dv}(t)=\mathbf{C}\left[e^{\mathbf{A}t}\mathbf{x}(0)+\int_0^t e^{-\mathbf{A}(t-\lambda)}\mathbf{Bv}(\lambda)d\lambda\right]+\mathbf{Dv}(t),\quad t\geq 0$$

$$=\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)+\int_0^t \mathbf{C}e^{-\mathbf{A}(t-\lambda)}\mathbf{Bv}(\lambda)d\lambda+\mathbf{Dv}(t),\quad t\geq 0$$

- **Using the definition of the unit impulse:**

$$\mathbf{y}(t)=\underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)}_{\mathbf{y}_{zi}(t)}+\underbrace{\int_0^t\left[\mathbf{C}e^{-\mathbf{A}(t-\lambda)}\mathbf{Bv}(\lambda)+\mathbf{D}\delta(t-\lambda)\mathbf{v}(\lambda)\right]d\lambda}_{\mathbf{y}_{zs}(t)},\quad t\geq 0$$

- **Recall our convolution integral for a single-input single-output system:**

$$y_{zs}(t)=h(t)*v(t)=\int_0^t h(t-\lambda)v(\lambda)d\lambda,\quad t\geq 0$$

- **Equating terms:**

$$\int_0^t\left[\mathbf{C}e^{-\mathbf{A}(t-\lambda)}\mathbf{B}v(\lambda)+\mathbf{D}\delta(t-\lambda)v(\lambda)\right]d\lambda=\int_0^t h(t-\lambda)v(\lambda)d\lambda$$

$$h(t)=\mathbf{C}e^{\mathbf{A}t}\mathbf{B}+\mathbf{D}\delta(t)$$

The impulse response can be computed directly from the coefficient matrices.

Solution via the Laplace Transform

- **Recall our state equations:**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t)$$

$$\mathbf{y}(t) = \mathbf{C}x(t) + \mathbf{D}\mathbf{v}(t)$$

- **Using the Laplace transform on the first equation:**

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{V}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{V}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{V}(s)$$

- **Comparing this to:**

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}(t-\lambda)}\mathbf{B}\mathbf{v}(\lambda)d\lambda, \quad t \geq 0$$

reveals that:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}$$

- **Continuing with the output equation:**

$$\mathbf{y}(t) = \mathbf{C}x(t) + \mathbf{D}\mathbf{v}(t)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{V}(s)$$

$$= (s\mathbf{I} - \mathbf{A})^{-1}x(0) + \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{V}(s)$$

The transfer function can be computed directly from the system parameters.

- **For zero initial conditions:**

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{X}(s) \quad \text{where} \quad \mathbf{H}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]$$

Summary

- Introduced the concept of a state variable.
- Described a linear system in terms of the general state equation.
- Demonstrated a process for deriving the state equations from a differential equation with a simple forcing function.
- Generalized this to an Nth-order differential equation with a more complex forcing function.
- Demonstrated these techniques on a 1^{st}-order (RC) and 2^{nd}-order (RLC) circuit.
- Observation: We have now encapsulated all passive circuit analysis (RLCs) into a single matrix equation. In fact, we now have a unified representation for all linear time-invariant systems.
- Introduced the time-domain and Laplace transform-based solutions to the state equations.
- Even nonlinear (and non-time-invariant) systems can be modeled using these techniques. However, the resulting differential equations are more complex. Fortunately, we have powerful numerical modeling techniques to handle such problems.