

UNIT-3

(Lecture-2)

Unit Step Response of a System

Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$(s + 2\zeta\omega_n)^2 \quad C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2} \quad \omega_n^2(1 - \zeta^2)$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, is the frequency of transient oscillations and is called **damped natural frequency**.
 - The inverse Laplace transform of above equation can be obtained easily if **C(s)** is written in the following form:
- $$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}} \omega_n \sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

- When $\zeta = 0$

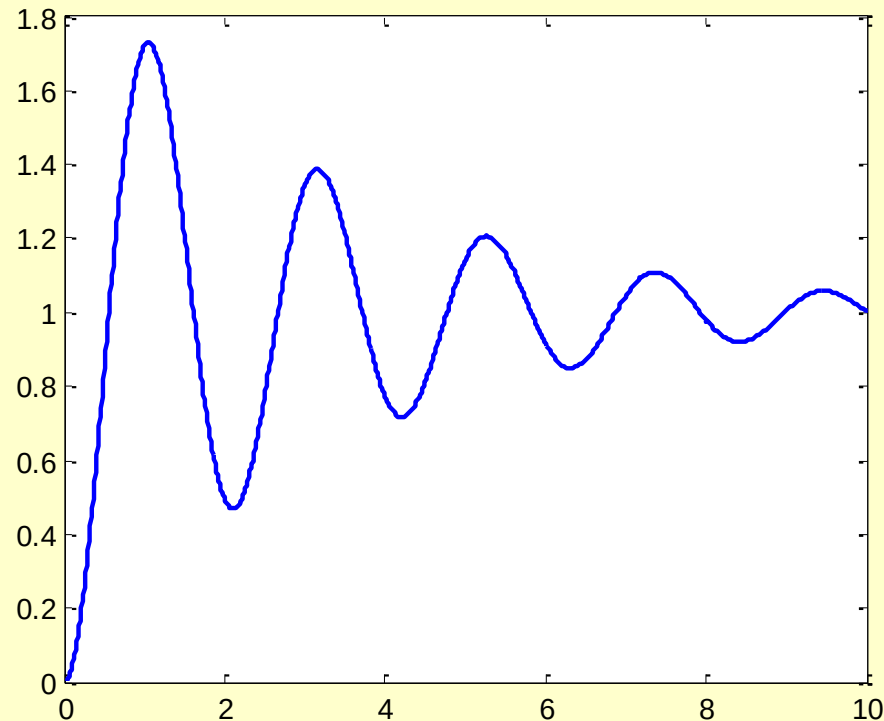
$$\begin{aligned}\omega_d &= \omega_n \sqrt{1-\zeta^2} \\ &= \omega_n\end{aligned}$$

$$c(t) = 1 - \cos \omega_n t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

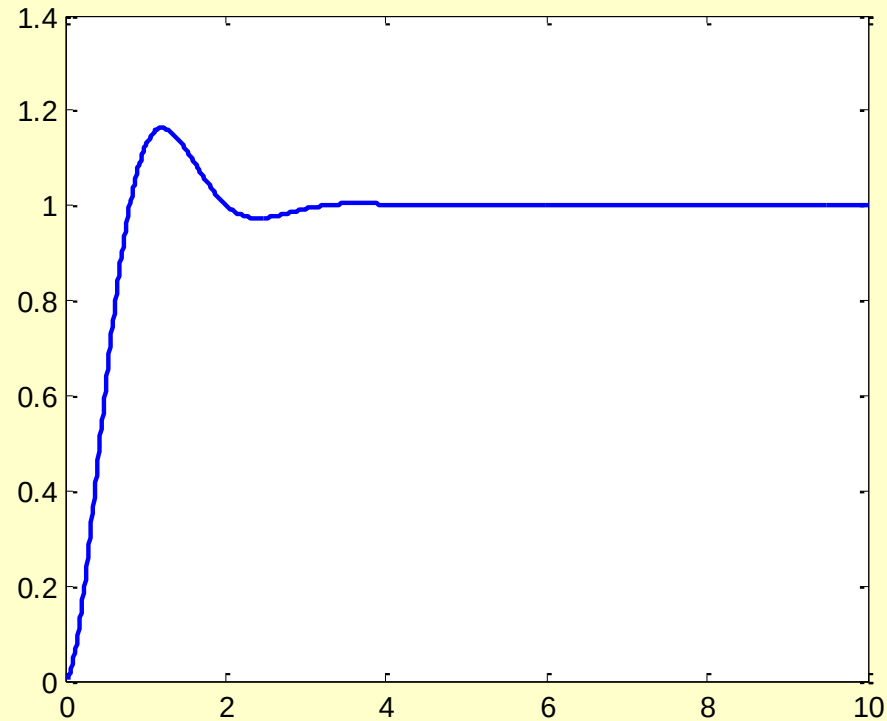
if $\zeta = 0.1$ and $\omega_n = 3$ rad/sec



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

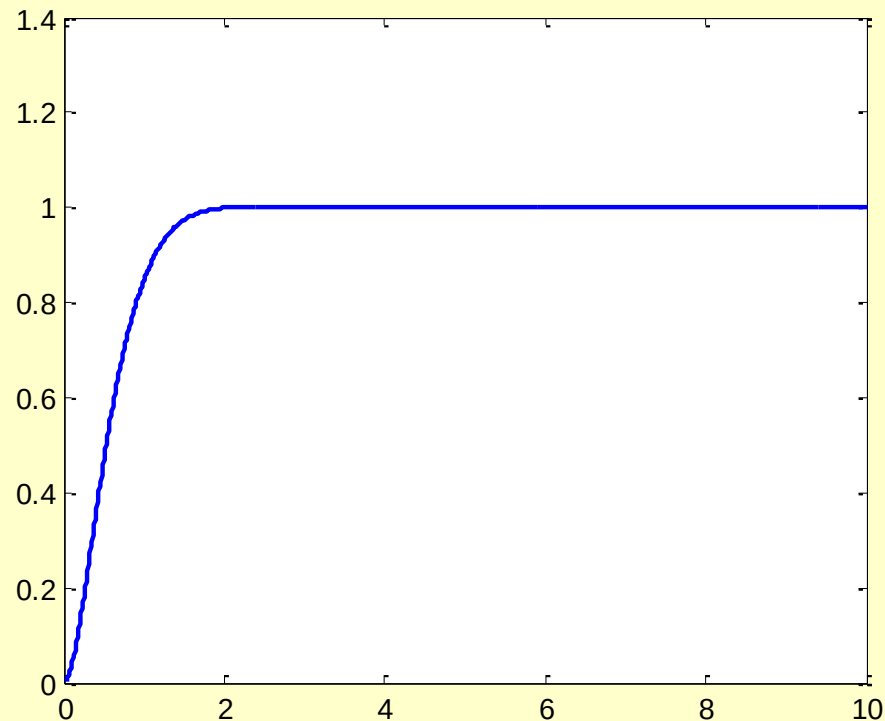
if $\zeta = 0.5$ and $\omega_n = 3 \text{ rad/sec}$



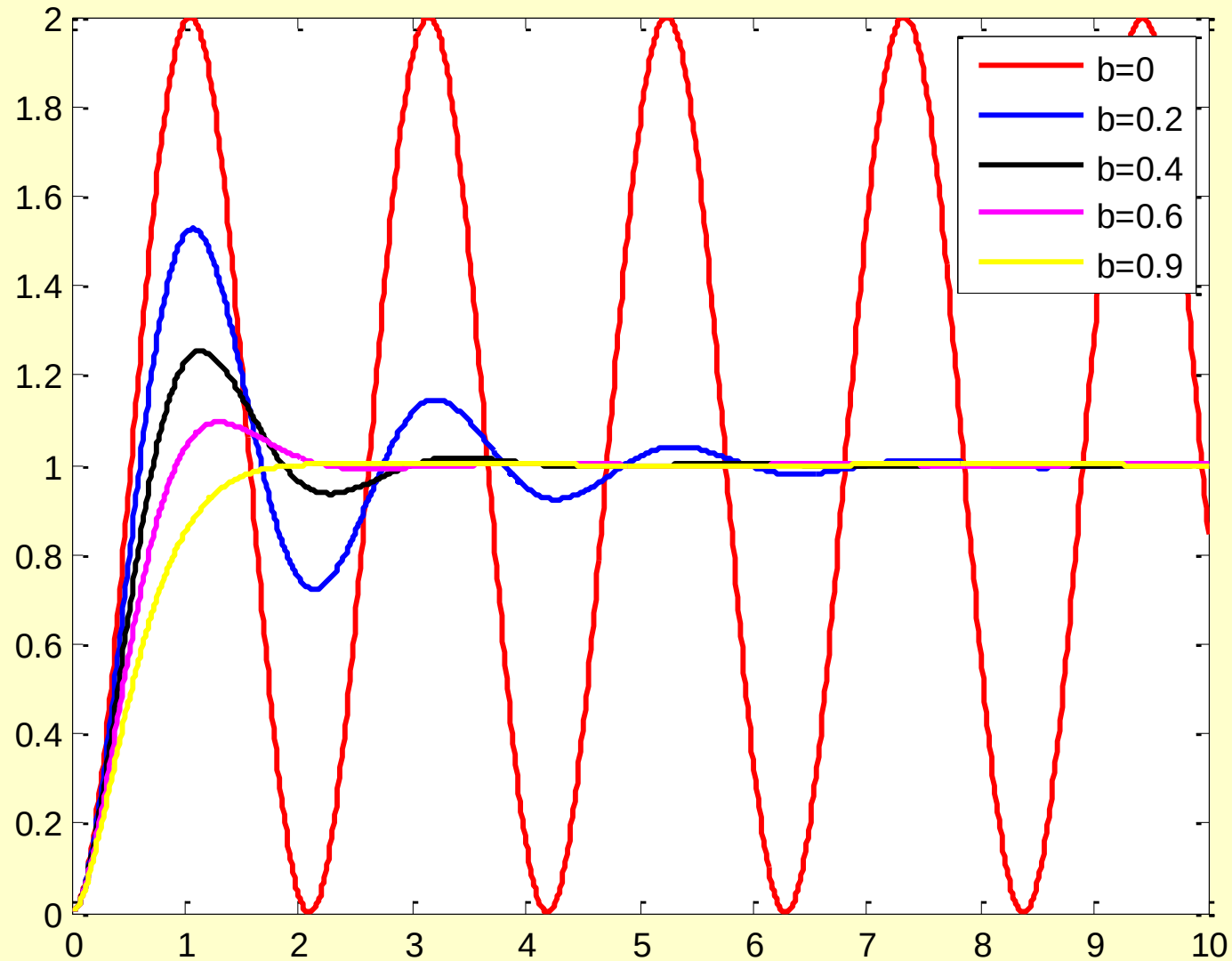
Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$

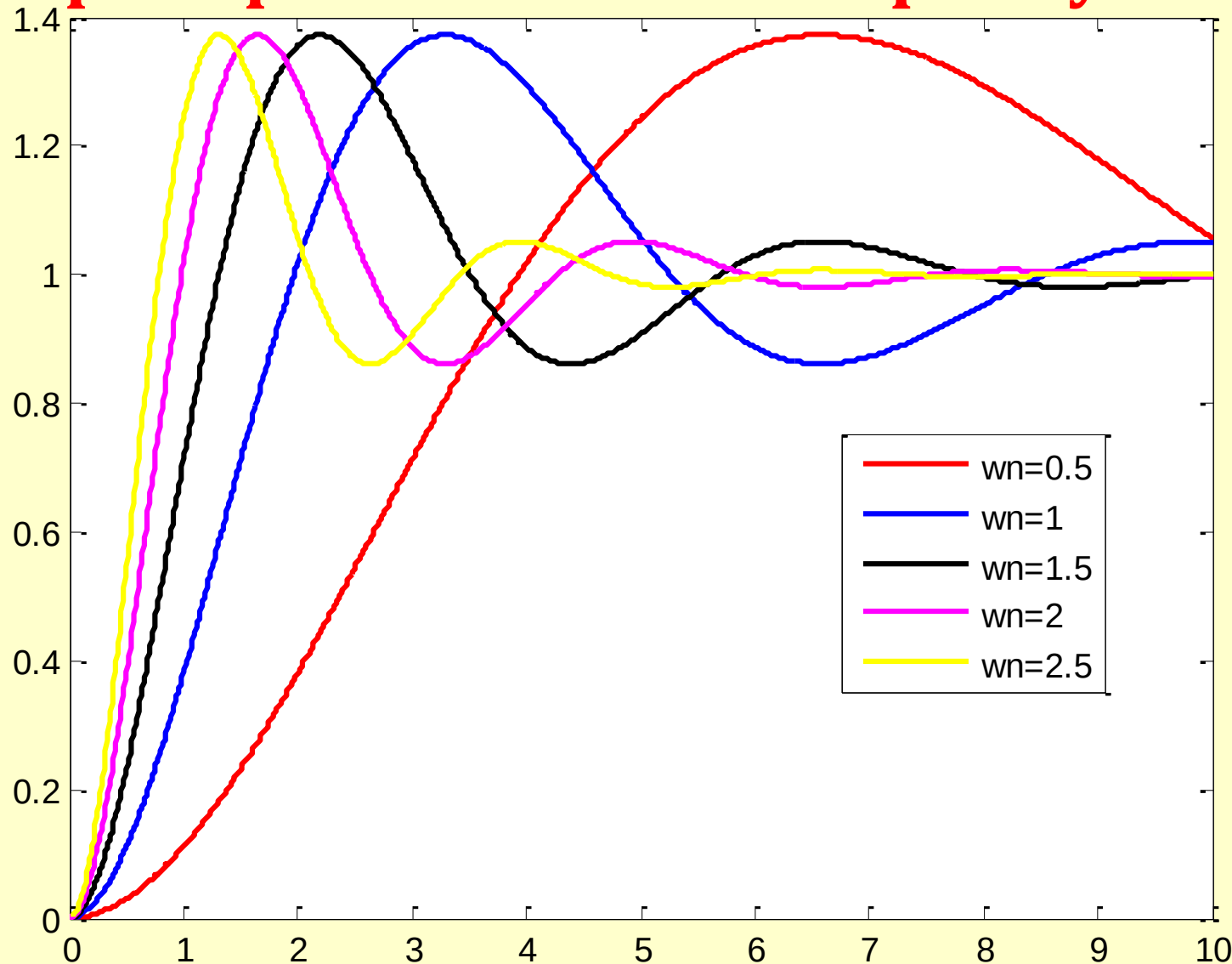
if $\zeta = 0.9$ and $\omega_n = 3 \text{ rad/sec}$



Step Response of underdamped System



Step Response of underdamped System



Step Response of critically damped System ($\zeta = 1$)

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \omega_n)^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

- The partial fraction expansion of above equation is given as

$$\frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t$$

$$c(t) = 1 - e^{-\omega_n t} (1 + \omega_n t)$$

System	Pole-zero Plot	Response
(a) $R(s) = \frac{1}{s} \rightarrow \boxed{\frac{G(s)}{s^2 + as + b}} \rightarrow C(s)$ General		
(b) $R(s) = \frac{1}{s} \rightarrow \boxed{\frac{9}{s^2 + 9s + 9}} \rightarrow C(s)$ Overdamped		
(c) $R(s) = \frac{1}{s} \rightarrow \boxed{\frac{9}{s^2 + 2s + 9}} \rightarrow C(s)$ Underdamped		
(d) $R(s) = \frac{1}{s} \rightarrow \boxed{\frac{9}{s^2 + 9}} \rightarrow C(s)$ Undamped		
(e) $R(s) = \frac{1}{s} \rightarrow \boxed{\frac{9}{s^2 + 6s + 9}} \rightarrow C(s)$ Critically damped		