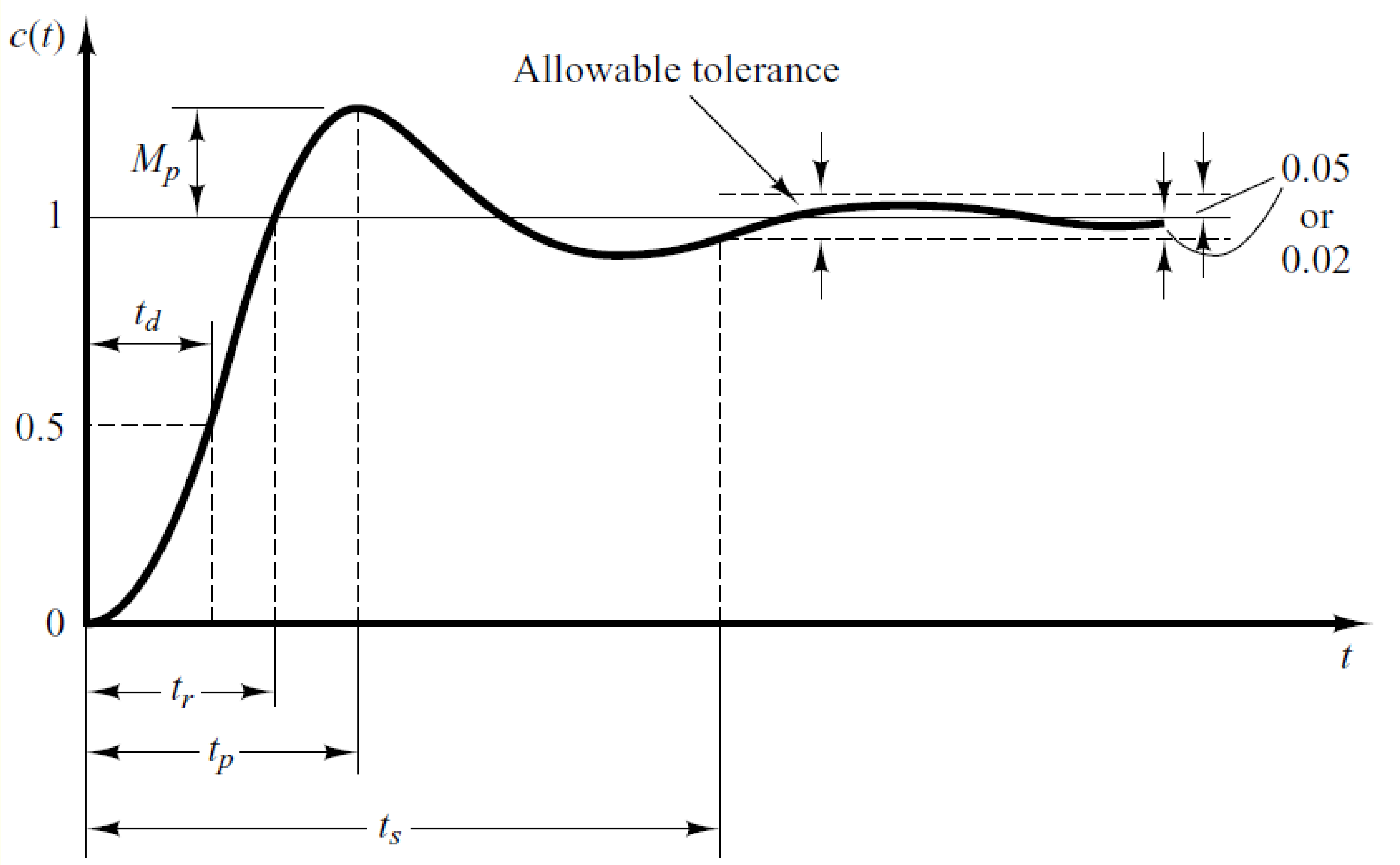
UNIT-3

(Lecture-4)

Time Domain Specifications of Underdamped System

Time Domain Specifications of Underdamped system



Time Domain Specifications (Rise Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

Put $t = t_r$ inaboveequatior

$$c(t_r) = 1 - e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

Where $c(t_r) = 1$

$$0 = -e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

$$-e^{-\zeta\omega_n t_r} \neq 0 \qquad 0 = \left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

Time Domain Specifications (Rise Time)

$$\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right] = 0$$

above equation can be re-writen as

$$\sin\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}\cos\omega_d t_r$$

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

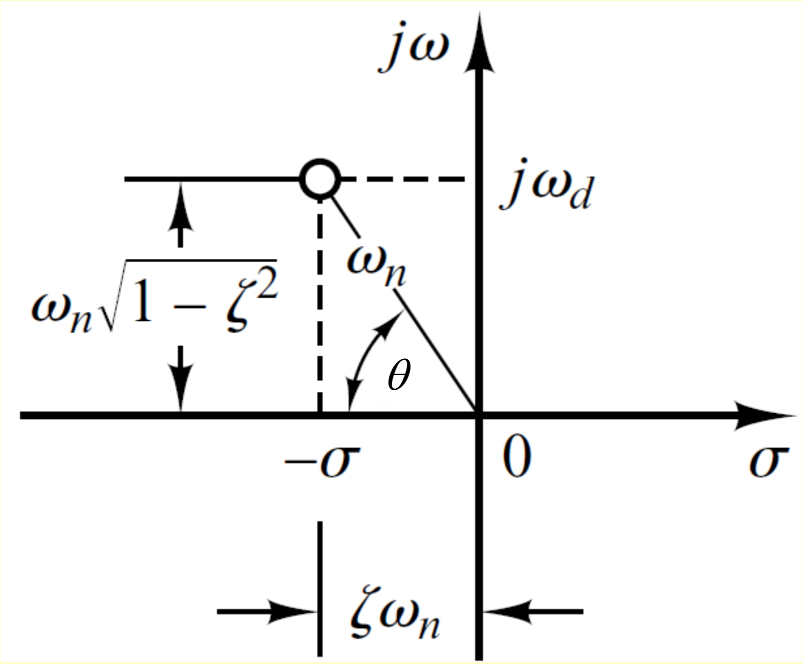
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Time Domain Specifications (Rise Time)

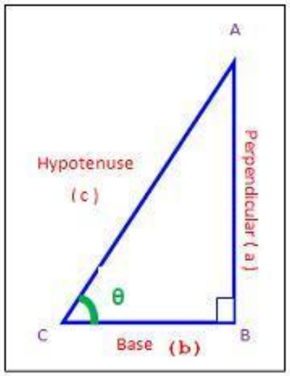
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta} \right)$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$



$$\theta = \tan^{-1}\frac{a}{b}$$



Time Domain Specifications (Peak Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- **In order to find peak time let us differentiate above equation w.r.t *t*.**

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] - e^{-\zeta\omega_n t}\left[-\omega_d\sin\omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\omega_n\cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\omega_n\cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d\sin\omega_d t - \frac{\zeta\omega_n\cancel{\sqrt{1-\zeta^2}}}{\cancel{\sqrt{1-\zeta^2}}}\cos\omega_d t\right]$$

Time Domain Specifications (Peak Time)

$$0 = e^{-\zeta\omega_n t}\left[\cancel{\zeta\omega_n \cos\omega_d t} + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t - \cancel{\frac{\zeta\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}\cos\omega_d t}\right]$$

$$e^{-\zeta\omega_n t}\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t\right] = 0$$

$$e^{-\zeta\omega_n t} \neq 0 \qquad \left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t\right] = 0$$

$$\sin\omega_d t\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right] = 0$$

Time Domain Specifications (Peak Time)

$$\sin\omega_d t\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}+\omega_d\right]=0$$

$$\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}+\omega_d\right]\neq 0$$

$$\sin\omega_d t=0$$

$$\omega_d t=\sin^{-1}0$$

$$t=\frac{0,\pi,2\pi,\cdots}{\omega_d}$$

- Since for underdamped stable systems first peak is maximum peak therefore,

$$\boxed{t_p=\frac{\pi}{\omega_d}}$$

Time Domain Specifications (Maximum Overshoot)

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$c(t_p) = 1 - e^{-\zeta\omega_n t_p}\left[\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right]$$

$$c(\infty) = 1$$

$$M_p = \left[\cancel{1} - e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right) - \cancel{1}\right]\times 100$$

Put $t_p = \dfrac{\pi}{\omega_d}$ in above equation

$$M_p = \left[- e^{-\zeta\omega_n \frac{\pi}{\omega_d}}\left(\cos\omega_d \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d \frac{\pi}{\omega_d}\right)\right]\times 100$$

Time Domain Specifications (Maximum Overshoot)

$$M_p = \left[-e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \left(\cos \cancel{\omega_d} \frac{\pi}{\cancel{\omega_d}} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \cancel{\omega_d} \frac{\pi}{\cancel{\omega_d}} \right) \right] \times 100$$

Put $\omega_d = \omega_n \sqrt{1\text{-}\zeta^2}$ in aboveequatior

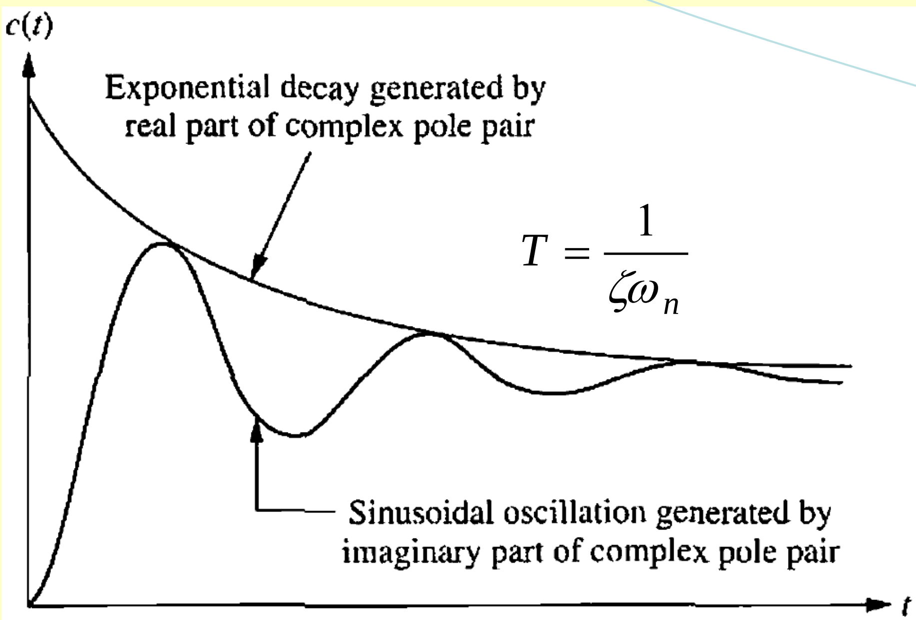
$$M_p = \left[-e^{-\zeta\cancel{\omega_n} \frac{\pi}{\cancel{\omega_n}\sqrt{1-\zeta^2}}} \left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \right] \times 100$$

$$M_p = \left[-e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \left(-1 + 0 \right) \right] \times 100$$

$$\boxed{M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100}$$

Time Domain Specifications (Settling Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$



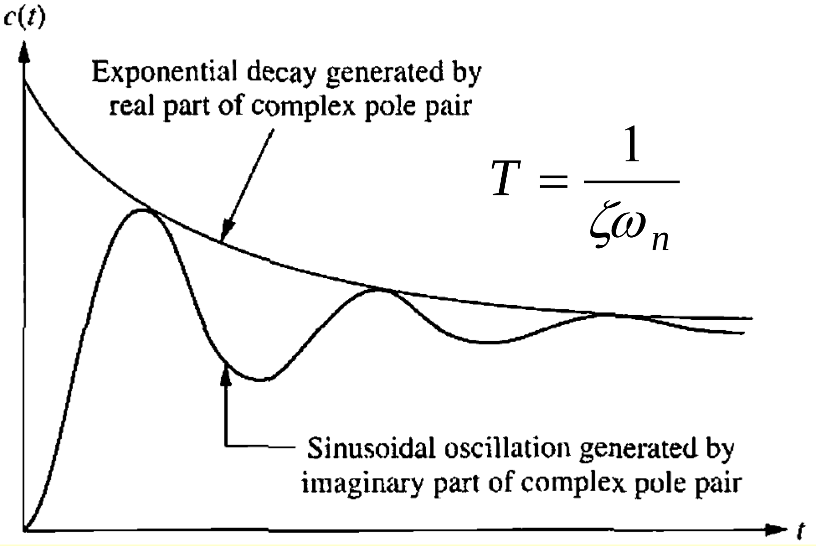
$$-\omega_n\zeta \pm \omega_n\sqrt{\zeta^2 - 1}$$

Real Part Imaginary Part

Time Domain Specifications (Settling Time)

- Settling time (2%) criterion
- Time consumed in exponential decay up to 98% of the input.

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$



- Settling time (5%) criterion
- Time consumed in exponential decay up to 95% of the input.

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Summary of Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta \omega_n}$$

Settling Time (4%)

$$t_s = 3T = \frac{3}{\zeta \omega_n}$$

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$