# PRINCIPLES OF COMMUNICATIONS

UNIT-2

LECTURE-2

## Properties of the $Z$-Transform

- Linearity:

$$a_1x_1[n] + a_2x_2[n] \longleftrightarrow a_1X_1(z) + a_2X_2(z), RoC = R_{x_1} \cap R_{x_2}$$

- Time Shifting Property:

$$x[n-n_0] \quad \longleftrightarrow \quad z^{-n_0}X(z),$$
$$RoC = R_x \ (except\ possible\ addition/deletion$$
$$of\ z=0\ or\ z=\infty)$$

- Exponential Weighting:

$$z_0^n x[n] \longleftrightarrow X(z_0^{-1}z), RoC = |z_0|R_x$$

  – The poles of the $Z$-transform are scaled by $|z_0|$

- Linear Weighting

$$nx(n) \longleftrightarrow -z\frac{dX(z)}{dz},$$

RoC = $R_x$ (except possible addition/deletion of $z = 0$ or $z = \infty$)

- Time Reversal

$$x[-n] \longleftrightarrow X(z^{-1}), \text{ RoC} = \frac{1}{R_x}$$

- Convolution

$$x[n] * y[n] \longleftrightarrow X(z)Y(z), \text{ RoC} = R_x \cap R_y$$

- Multiplication

$$x[n]w[n] \longleftrightarrow \frac{1}{2\pi j}\int X(v)w\left(\frac{z}{v}\right)v^{-1}dv$$

# Inverse Z-Transform Examples

- Using long division: Causal sequence

$$\frac{1}{1 - az^{-1}}, \text{RoC} = |z| > |a| \quad = \quad 1 + az^{-1} + az^{-2} + az^{-3} + \cdots$$

$$\text{IZT}(1 + az^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots) = a^{n}u[n]$$

- Using long division: Noncausal sequence

$$\frac{1}{1 - az^{-1}}, \text{RoC} = |z| < |a|$$

Here the IZT is computed as follows:

$$\text{IZT}\left(\frac{1}{1 - az^{-1}}\right) = \text{IZT}\left(\frac{\quad}{-a + z}\right)$$

This results in:

$$\text{IZT}(-a^{-1}z + a^{-2}z^{2} + a^{-3}z^{3} + \cdots) = -a^{n}u[-n - 1]$$

- Inverse $Z$-transform - using Power series expansion

$$X(z) = log(1 + az^{-1}), |z| > |a|$$

Using the Power Series expansion for $\log(1 + x)$, $|x| < 1$, we have

$$X(z) = \sum_{1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

The IZT is given by

$$\begin{aligned} x[n] &= \frac{(-1)^{n+1} a^n}{n}, n \geq 1 \\ &= 0, n \leq 0 \end{aligned}$$

- Inverse $Z$-transform - Inspection method

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}, |z| > |a|$$

$$Given X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, |z| > |\frac{1}{2}|$$

$$\Longrightarrow x[n] = (\frac{1}{2})^n u[n]$$

- Inverse $Z$-transform - Partial fraction method
  - Example 1: All-Pole system

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{6}z^{-1})}, |z| > \frac{1}{3}$$

Using partial fraction method, we have:

$$
\begin{aligned}
X(z) &= \frac{A_1}{1-\frac{1}{3}z^{-1}} + \frac{A_2}{1-\frac{1}{6}z^{-1}}, \\
& \quad |z| > \frac{1}{3} \\
A_1 &= (1-\frac{1}{3}z^{-1})X(z)|_{z=\frac{1}{3}} \\
A_2 &= (1-\frac{1}{6}z^{-1})X(z)|_{z=\frac{1}{6}} \\
A_1 &= 2 \\
A_2 &= -1 \\
x(n) &= 2(\frac{1}{3})^n u[n] - 1(\frac{1}{6})^n u[n]
\end{aligned}
$$

– Example 2: Pole-Zero system

$$\begin{aligned}
X(z) &= \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, |z|>1 \\
&- \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \\
&= 2+\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \\
&= 2-\frac{9}{1-\frac{1}{2}z^{-1}}+\frac{8}{1-z^{-1}} \\
x[n] &= 2\delta[n]-9(\frac{1}{2})^n u[n]+8u[n]
\end{aligned}$$

– Example 3: Finite length sequences

$$\begin{aligned} X(z) &= z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1}) \\ &= z^2 - frac12z - 1 + \frac{1}{2}z^{-1} \\ &= \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1] \end{aligned}$$

## Inverse $Z$-Transform Problem

1. Given $X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$, determine all the possible sequences for $x[n]$.

Hint: Remember that the RoC must be a continuous region