PRINCIPLES OF COMMUNICATIONS

UNIT-2

LECTURE-7

Random Process through a linear filter

- A random process $X(t)$ is applied as input to a linear time-invariant filter of impulse response $h(t)$,
- It produces a random process $Y(t)$ at the filter output as shown in Figure 1

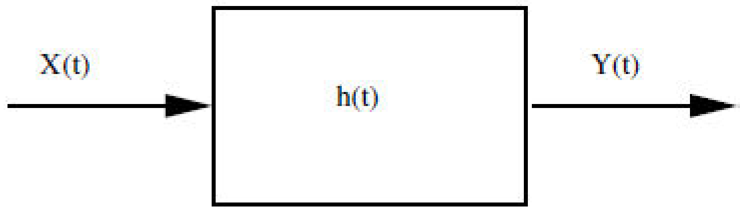

Figure 1: Transmission of a random process through a linear filter

- Difficult to describe the probability distribution of the output random process $Y(t)$, even when the probability distribution of the input random process $X(t)$ is completely specified for $-\infty \leq t \leq +\infty$.

- Estimate characteristics like mean and autocorrelation of the output and try to analyse its behaviour.
- Mean The input to the above system X(t) is assumed stationary. The mean of the output random process Y (t) can be calculated

$$
\begin{aligned}
m_Y(t) &= E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(\tau)X(t-\tau)\,d\tau\right] \\
&= \int_{-\infty}^{+\infty} h(\tau)E[X(t-\tau)]\,d\tau \\
&= \int_{-\infty}^{+\infty} h(\tau)m_X(t-\tau)\,d\tau \\
&= m_X \int_{-\infty}^{+\infty} h(\tau)\,d\tau \\
&= m_X H(0)
\end{aligned}
$$

where H(0) is the zero frequency response of the system.

- Autocorrelation The autocorrelation function of the output random process Y (t). By definition, we have

$$R_Y(t, u) = E[Y(t)Y(u)]$$

where t and u denote the time instants at which the process is observed. We may therefore use the convolution integral to write

$$R_Y(t,u) = E\left[\int_{-\infty}^{+\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{+\infty} h(\tau_2)X(t-\tau_2)\,d\tau_2\right]$$

$$= \int_{-\infty}^{+\infty} h(\tau_1)\,d\tau_1 \int_{-\infty}^{+\infty} h(\tau_2)E\left[X(t-\tau_1)X(t-\tau_2)\right]d\tau_2$$

- When the input X(t) is a wide-stationary random process,
  - The autocorrelation function of X(t) is only a function of the difference between the observation times $t - \tau_1$ and

$u - \tau_2$.

– Putting $\tau = t - u$, we get

$$R_Y(\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)\, d\tau_1\, d\tau_2$$

– $R_Y(0) = E[Y^2(t)]$

– The mean square value of the output random process $Y(t)$ is obtained by putting $\tau = 0$ in the above equation.

$$\begin{aligned} E[Y^2(t)] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\tau_1)h(\tau_2)R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} H(\omega) \exp(j\omega\tau_1)\, d\omega \right] h(\tau_2) R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega)\, d\omega \int_{-\infty}^{+\infty} h(\tau_2)\, d\tau_2 \int_{-\infty}^{+\infty} R_X(\tau_2 - \tau_1) \exp(j2\omega\tau_1)\, d\tau_1 \end{aligned}$$

– Putting $\tau = \tau_2 - \tau_1$

$$
\begin{aligned}
E[Y^2(t)] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)\,d\omega \int_{-\infty}^{+\infty} h(\tau_2)\exp(j\omega\tau_2)\,d\tau_2 \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j2\omega\tau)\,d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)\,d\omega \int_{-\infty}^{+\infty} H^*(\omega)\,d\omega \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j\omega\tau)\,d\tau
\end{aligned}
$$

– This is simply the Fourier Transform of the autocorrelation function $R_X(t)$ of the input random process $X(t)$. Let this transform be denoted by $S_X(f)$.

$$
S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau)\exp(-j\omega\tau)\,d\tau
$$

– $S_X(\omega)$ is called the *power spectral density* or *power spectrum* of the wide-sense stationary random process $X(t)$.

$$
E[Y^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2 S_X(\omega)\,df
$$

- *The mean square value of the output of a stable linear time-invariant filter in response to a wide-sense stationary random process is equal to the integral over all frequencies of the power spectral density of the input random process multiplied by the squared magnitude of the transfer function of the filter.*