Orbit Determination

(1) Find state-vector (position and velocity) at time $t=t_0$

(2) Determine orbit from all types of observations

(1) Gives Kepler elements and time derivatives from $J_2=-C_{20}$

(2) –Short arcs very precise (3.3.1)

-Long arcs for ”prediction” (3.3.2)

Kepler orbits

$$E = \overline{M} + (e - \frac{1}{8}e^3 + \frac{1}{192}e^5 - \frac{1}{9216}e^7)\sin\overline{M}$$

$\overline{M}$ = Mean anomaly

$$\tan\nu = (\tan \mathrm{f})_{\mathrm{Kaula}} = \frac{\sqrt{1-e^e}\sin E}{\cos E - e}$$

Non-linear system of equations for all obs.:

$$\begin{pmatrix} X_s \\ Y_s \\ Z_s \end{pmatrix} = r \begin{pmatrix} \cos(\nu+\omega)\cos\Omega - \sin(\nu+\omega)\sin\Omega\cos i \\ \cos(\nu+\omega)\sin\Omega + \sin(\nu+\omega)\cos\Omega\cos i \\ \sin(\nu+\omega)\sin i \end{pmatrix},$$

$$r = \frac{a(1-e^2)}{1+e\cos\nu} = a\,(1-e\cos E).$$

Linearisation using Kepler elem.

Start values: a,e,f+ω,Ω,i

$$\text{Taylor development:}$$

$$X_S(t_1) = X_S(a_0, e_0, \omega + f_0, \Omega_0, i_0) +$$

$$\frac{\partial X_S}{\partial a}\Delta a + \cdots + \frac{\partial X_S}{\partial i}\Delta i$$

- $$Y_S(t_1) = \\ . \\ Z_S(t_1) =$$

General orbit determination

Analytic orbit determination uses knowledge of C_{ij} to compute

$$\Omega(t) = \Omega_0(t_0) + \frac{\partial \Omega}{\partial t}\Big|_{t_0} \cdot (t - t_0)$$

and similar for 5 other elements

More difficult for drag, solar pressure etc.

Start values describe reference Kepler orbit:

$$\Omega_0(t_0), i_0(t_0), a_0(t_0), \quad \text{etc.}$$

- Truncated series are used, so limited precision.
- 500 terms give 1 m.

Numerical integration

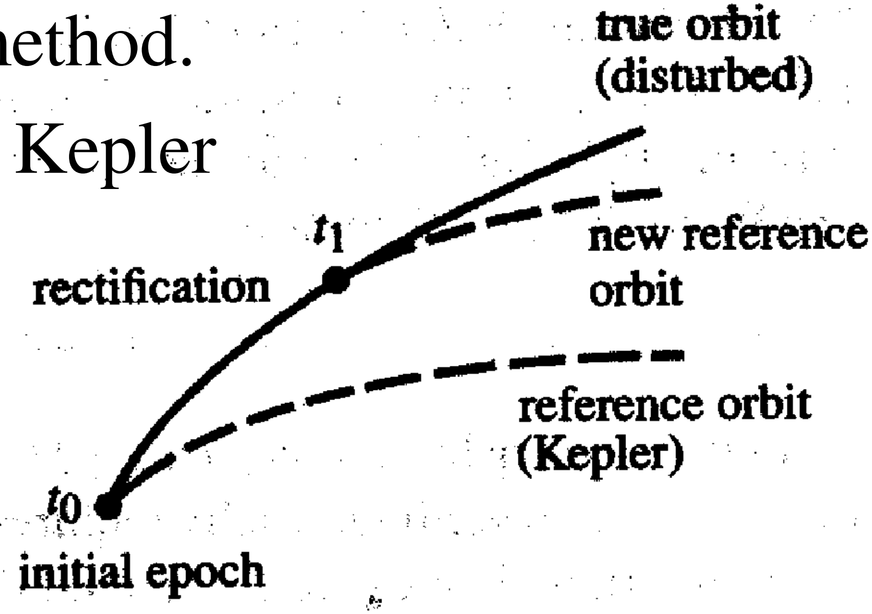
Equations of motion : 2 sets of 1.- order equations :

$$\dot{\vec{r}} = \vec{v}, \quad \dot{\vec{v}} = \vec{k}_S - \frac{GM}{r^3}\vec{r}, \quad \vec{k}_S \text{ all perturbing forces}$$

- Cartesian coordinates not optimal. Spherical better (r,θ,Φ).Steps of numerical integration smaller.
- Cowell (1910) method.
- Encke, 1857: use Kepler

orbit as reference:

Osculating orbit.



Enckes method

$$\ddot{\vec{r}} + \frac{GM}{r^3}\vec{r} = \vec{k}_S$$

$$\ddot{\vec{\rho}} + \frac{GM}{r^3}\vec{\rho} = 0$$

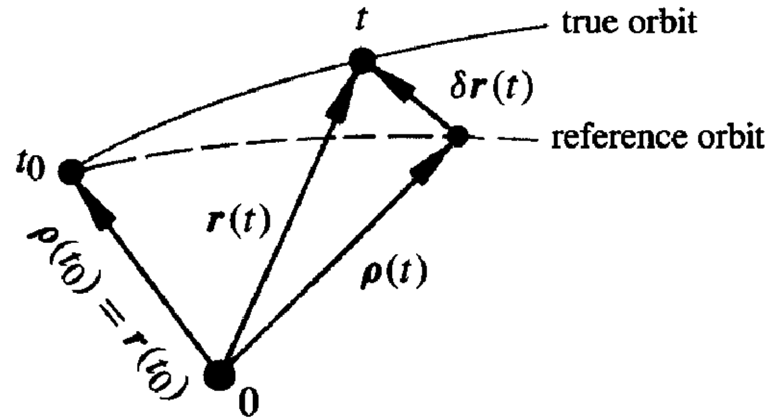
$$t = t_0,\ \text{then}\ \vec{r}(t_0) = \vec{\rho}(t_0),$$

$$\dot{\vec{r}}(t_0) = \dot{\vec{\rho}}(t_0),\ \delta\vec{r} = \vec{r} - \vec{\rho}$$

$$\delta\ddot{\vec{r}} = \vec{k}_S + \frac{GM}{\rho^3}((1 - \frac{\rho^3}{r_3})\vec{r} - \delta\vec{r})$$

$$\delta\vec{r}(t_0 - \Delta t)\ \text{from numerical integration}$$

Runge - Kutta's method



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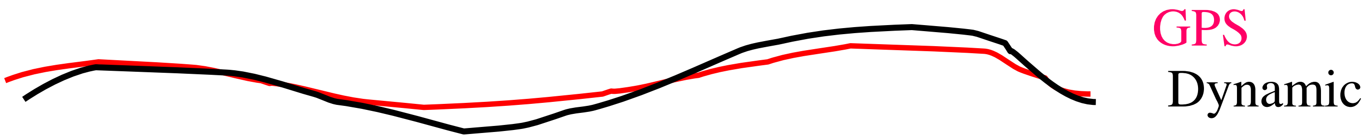
Orbit determination using GPS

- POD= Precise Orbit Determination
- Dynamic:
- (a) orbit determined using integration of qeuations of motion
- (b) adjusted to GPS measurements
- Kinematic:

- From GPS measurements
- Reduced Dynamic, like Dynamic byt GPS data adjusted using Kalman filter



Orbit representation

(1) Kepler elements and linear pertubations

Transit (Doppler) $\dot{\Omega}, \dot{\omega}$

Corrections: every minute, along track, cross-track and radially

GPS: Every hour: $\dot{\Omega}, \dot{\omega}, \dot{i}$ and $\cos, \sin$ of $f + \omega, i, r$

(2) Polynomial representation: (only 1 – 2 revolutions

$$F(x) = \sum_{n=0}^{m} a_n \varphi_n, \quad \sin, \cos, \text{Chebyshev},$$

Orthogonal functions on $[0,1]$

$$TRANSIT : \{1, t, \sin \pi t, \cos 2\pi t\}$$

Chebychev:

$$
\begin{aligned}
T_0(\tau) &= 1, \\
T_1(\tau) &= \tau, \\
T_n(\tau) &= 2\tau T_{n-1}(\tau) - T_{n-2}(\tau); |\tau| \le 1, \quad n \ge 2; \\
T_1'(\tau) &= \frac{d\tau}{dt} = \dot{\tau}, \\
T_2'(\tau) &= 4\tau\dot{\tau}, \\
T_n'(\tau) &= \frac{2n}{n-1}\tau T_{n-1}'(\tau) - \frac{n}{n-2}T_{n-2}'(\tau); \quad n \ge 3, \\
T_2''(\tau) &= 4(\dot{\tau})^2, \\
T_3''(\tau) &= 24\tau(\dot{\tau})^2, \\
T_n''(\tau) &= \frac{2n}{n-1}(\dot{\tau}T_{n-1}'(\tau) + \tau T_{n-1}''(\tau)) - \frac{n}{n-2}T_{n-2}''(\tau), \quad n \ge 4.
\end{aligned}
$$

Simplified short-arc repr.

∇V computed in spherical coordinates from

$$V(\bar{\varphi}, \lambda, r) = \frac{GM}{r}\left(1 - \sum_n \sum_m \left(\frac{a_e}{r}\right)^n \bar{Y}_{nm}(\bar{\varphi}, \lambda) \cdot \bar{C}_{nm}\right)$$

We also consider Earth rotation, so

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial V}{\partial x} + 2\omega\dot{x} + \omega^2 x \\ \frac{\partial V}{\partial y} + 2\omega\dot{y} + \omega^2 y \\ \frac{\partial V}{\partial z} \end{Bmatrix}$$

Orbit selection

- How frequently must the satellite cross Equator ?

$$v_c = \sqrt{\frac{GM}{r}},\ \text{circular orbit}$$

$$\text{Revolution time}: U = 84.491\left(\frac{r}{r_{e-mean}}\right)^{\frac{3}{2}}\ \text{(minutes)}$$

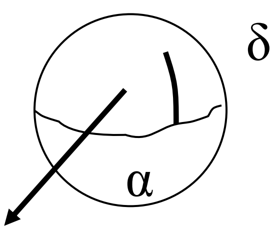
- Where is the ground-track ? (spherical earth)
- .

$$u = \omega + f$$

$$\alpha = \Omega + \tan^{-1}(\cos i \tan u)$$

$$\delta = \sin^{-1}(\sin i \sin u)$$

$$\lambda = \alpha - \dot{\theta}\ \text{(GST)}, \varphi = \delta$$



Sun-syncroneous, or geostationary

$$\frac{d\Omega}{dt} = 0.^{0}9863 / day$$

$$\frac{d\Omega}{dt} \approx C_{20} \frac{3na_e^3}{2a^2(1-e^2)^2} \cos i$$

$$\text{Two possibilities : change a or i !}$$

- · $\text{Example : a} = 1000\text{km}, \text{i} = 99.47^{0}$

$$\text{Geostationary : a} = 42165 \text{ km}$$

Bringing the satellite in orbit: Transfer orbit

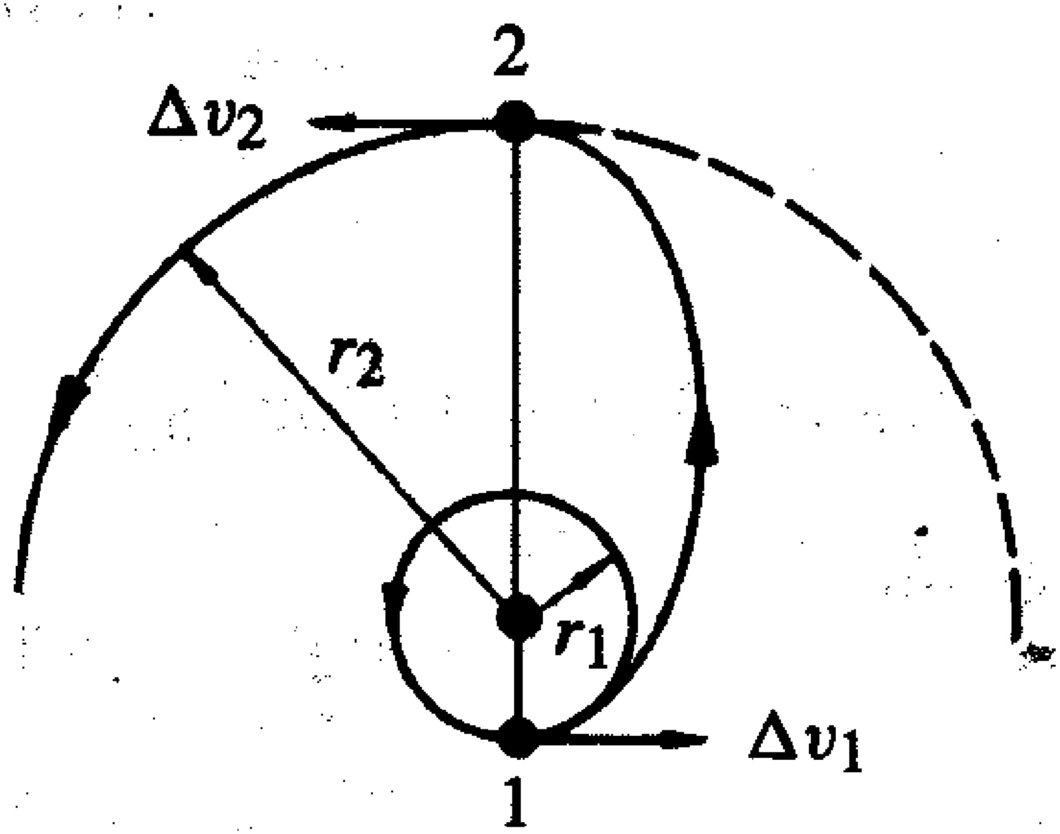


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Figure 3.28. Hohmann transfer orbit

Transfer orbit, velocity requirement

$2a_{transfer} = r_1 + r_2$ with total energy

$$E_t = -\frac{GM}{2a} = -\frac{GM}{r_1 + r_2}, \quad \text{so:}$$

$$v_1 = \sqrt{2\left(\frac{GM}{r_1} + E_t\right)}, \quad v_{c_1} = \sqrt{\frac{GM}{r_2}} \text{ (circular orb.)}$$

From circle to ellipse: increate in velocity

$\Delta \mathrm{v}_1 = v_1 - v_{c_1}$, and then: $\Delta v_2 = v_{c_2} - v_2$

$$\Delta \mathrm{v}_1 = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2/r_1}{1 + r_2/r_1}} - 1\right), \Delta \mathrm{v}_2 = \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2}{1 + r_2/r_1}}\right)$$

Lagrange points

- Stable points in Sun, Earth, Moon system: