

# Fourier Series

## **Complex Form of the Fourier Series**

# Complex Exponentials

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j \sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2} \left( e^{jn\omega_0 t} + e^{-jn\omega_0 t} \right)$$

$$\sin n\omega_0 t = \frac{1}{2j} \left( e^{jn\omega_0 t} - e^{-jn\omega_0 t} \right) = -\frac{j}{2} \left( e^{jn\omega_0 t} - e^{-jn\omega_0 t} \right)$$

# Complex Form of the Fourier Series

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \\ &= \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} a_n (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) - \frac{j}{2} \sum_{n=1}^{\infty} b_n (e^{jn\omega_0 t} - e^{-jn\omega_0 t}) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{1}{2} (a_n - jb_n) e^{jn\omega_0 t} + \frac{1}{2} (a_n + jb_n) e^{-jn\omega_0 t} \right] \\ &= c_0 + \sum_{n=1}^{\infty} [c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}] \end{aligned}$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2} (a_n - jb_n)$$

$$c_{-n} = \frac{1}{2} (a_n + jb_n)$$

# Complex Form of the Fourier Series

$$\begin{aligned} f(t) &= c_0 + \sum_{n=1}^{\infty} \left[ c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} \right] \\ &= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \end{aligned}$$

$$\begin{aligned} c_0 &= \frac{a_0}{2} \\ c_n &= \frac{1}{2}(a_n - jb_n) \\ c_{-n} &= \frac{1}{2}(a_n + jb_n) \end{aligned}$$

# Complex Form of the Fourier Series

$$c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt$$

$$c_n = \frac{1}{2} (a_n - jb_n)$$

$$= \frac{1}{T} \left[ \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t dt - j \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t dt \right]$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) (\cos n\omega_0 t - j \sin n\omega_0 t) dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

$$c_{-n} = \frac{1}{2} (a_n + jb_n) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{jn\omega_0 t} dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2} (a_n - jb_n)$$

$$c_{-n} = \frac{1}{2} (a_n + jb_n)$$

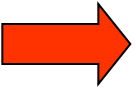
# Complex Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

$$c_0 = \frac{a_0}{2}$$
$$c_n = \frac{1}{2}(a_n - jb_n)$$
$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

If  $f(t)$  is real,

  $c_{-n} = c_n^*$

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$|c_n| = |c_{-n}| = \frac{1}{2} \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1} \left( -\frac{b_n}{a_n} \right)$$

$$n = \pm 1, \pm 2, \pm 3, \dots$$

$$c_0 = \frac{1}{2} a_0$$

# Complex Frequency Spectra

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$|c_n| = |c_{-n}| = \frac{1}{2} \sqrt{a_n^2 + b_n^2} \qquad \phi_n = \tan^{-1} \left( -\frac{b_n}{a_n} \right) \quad n = \pm 1, \pm 2, \pm 3, \dots$$

$$c_0 = \frac{1}{2} a_0$$

