

# Decomposition

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t dt \quad n = 1, 2, \dots$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t dt \quad n = 1, 2, \dots$$

# Proof

Use the following facts:

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t) dt = 0, \quad m \neq 0$$

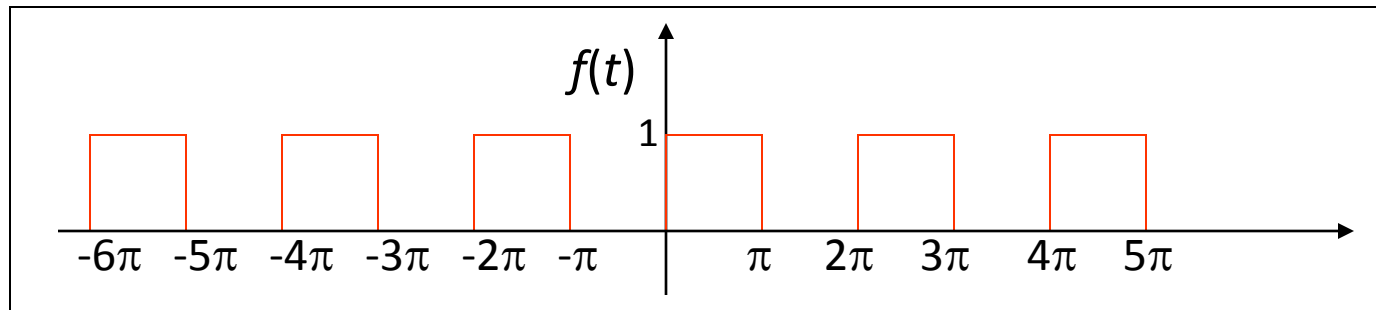
$$\int_{-T/2}^{T/2} \sin(m\omega_0 t) dt = 0, \quad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t) \cos(n\omega_0 t) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t) \sin(n\omega_0 t) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t) \cos(n\omega_0 t) dt = 0, \quad \text{for all } m \text{ and } n$$

# Example (Square Wave)



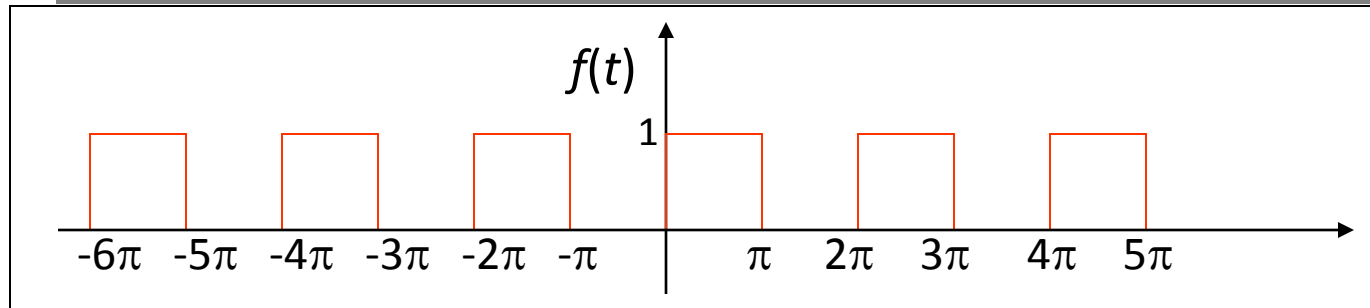
$$a_0 = \frac{2}{2\pi} \int_0^\pi 1 dt = 1$$

$$a_n = \frac{2}{2\pi} \int_0^\pi \cos ntdt = \frac{1}{n\pi} \sin nt \Big|_0^\pi = 0 \quad n = 1, 2, \dots$$

$$b_n = \frac{2}{2\pi} \int_0^\pi \sin ntdt = -\frac{1}{n\pi} \cos nt \Big|_0^\pi = -\frac{1}{n\pi} (\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1, 3, 5, \dots \\ 0 & n = 2, 4, 6, \dots \end{cases}$$

# Example (Square Wave)

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \left( \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \dots \right)$$

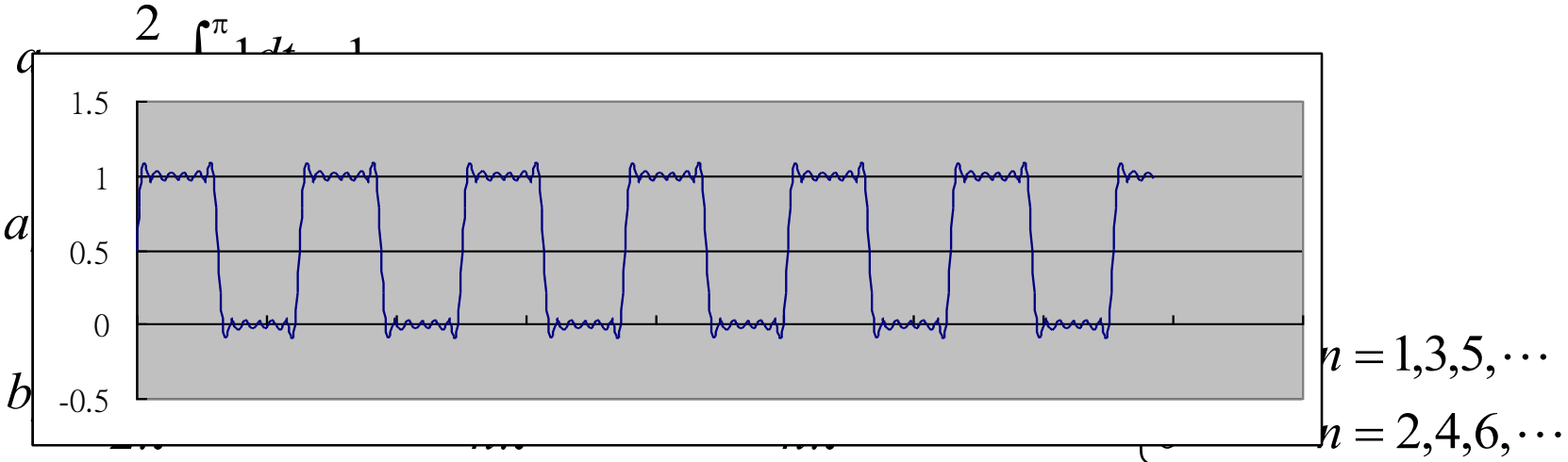
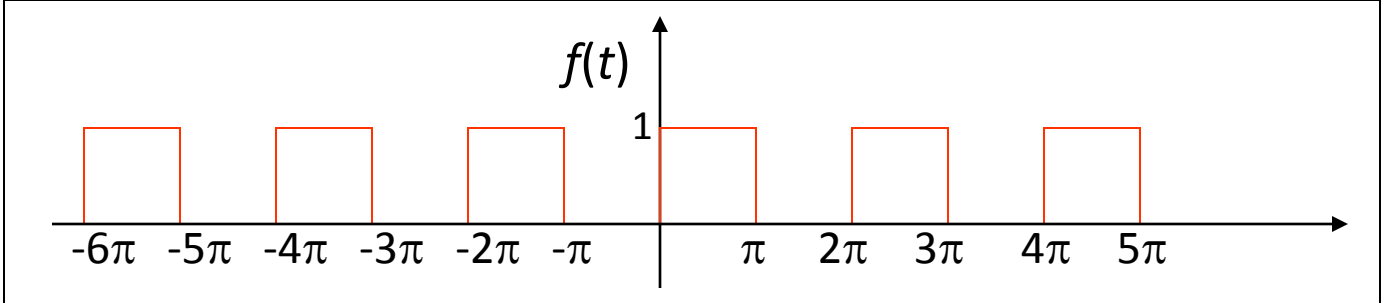


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$$f(t) = \frac{1}{2} + \frac{2}{\pi} \left( \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \dots \right)$$



# Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

  
DC Part

  
Even Part

  
Odd Part

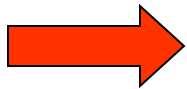
  
T is a period of all the above signals

# Harmonics

Define  $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$  called the *fundamental angular frequency*.

Define  $\omega_n = n\omega_0$  called the *n-th harmonic* of the periodic function.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

# Harmonics

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left( \frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \omega_n t \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} (\cos \theta_n \cos \omega_n t + \sin \theta_n \sin \omega_n t) \\ &= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n) \end{aligned}$$



# Amplitudes and Phase Angles

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

harmonic amplitude

phase angle

$$C_0 = \frac{a_0}{2}$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$