Inverse Laplace Transform

The Laplace Transform

Time Differentiation:

We can extend the previous to show;

$$L\left[\frac{df(t)^2}{dt^2}\right] = s^2 F(s) - sf(0) - f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

general case

$$L\left[\frac{df(t)^n}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- \ldots - f^{(n-1)}(0)$$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-st}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
te^{-at}	$\dfrac{1}{(s+a)^2}$
$t^n e^{-at}$	$\dfrac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\dfrac{w}{s^2+w^2}$
$\cos(wt)$	$\dfrac{s}{s^2+w^2}$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
$e^{-at}\sin(wt)$	$\dfrac{w}{(s+a)^2+w^2}$
$e^{-at}\cos(wt)$	$\dfrac{s+a}{(s+a)^2+w^2}$
$\sin(wt+\theta)$	$\dfrac{s\sin\theta+w\cos\theta}{s^2+w^2}$
$\cos(wt+\theta)$	$\dfrac{s\cos\theta-w\sin\theta}{s^2+w^2}$

Yes !

The Laplace Transform

Common Transform Properties:

f(t)	F(s)
$f(t-t_0)u(t-t_0), t_0 \ge 0$	$e^{-t_o s}F(s)$
$f(t)u(t-t_0), t \ge 0$	$e^{-t_o s}L[f(t+t_0)$
$e^{-at}f(t)$	$F(s+a)$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - s^0 f^{n-1}f(0)$
$tf(t)$	$-\frac{dF(s)}{ds}$
$\int_0^t f(\lambda)d\lambda$	$\frac{1}{s}F(s)$

The Laplace Transform

Using Matlab with Laplace transform:

Example

Use Matlab to find the transform of te^{-4t}

The following is written in italic to indicate Matlab code

```
syms t,s
laplace(t*exp(-4*t),t,s)
ans =
      1/(s+4)^2
```

The Laplace Transform

Using Matlab with Laplace transform:

Example

Use Matlab to find the inverse transform of

$$F(s) = \frac{s(s+6)}{(s+3)(s^2+6s+18)} \qquad prob.12.19$$

```
syms s t

ilaplace(s*(s+6)/((s+3)*(s^2+6*s+18)))

ans =
     -exp(-3*t)+2*exp(-3*t)*cos(3*t)
```

The Laplace Transform

Theorem: Initial Value Theorem:

If the function f(t) and its first derivative are Laplace transformable and f(t) Has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)$$

Initial Value Theorem

The utility of this theorem lies in not having to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

The Laplace Transform

Example: Initial Value Theorem:

Given;

$$F(s) = \frac{(s+2)}{(s+1)^2 + 5^2}$$

Find f(0)

$$f(0) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\frac{(s+2)}{(s+1)^2 + 5^2} = \lim_{s \to \infty}\left[\frac{s^2 + 2s}{s^2 + 2s + 1 + 25}\right]$$

$$= \lim_{s \to \infty} \frac{s^2/s^2 + 2s/s^2}{s^2/s^2 + 2s/s^2 + (26/s^2)} = 1$$

The Laplace Transform

Theorem: Final Value Theorem:

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)$$

Final Value Theorem

Again, the utility of this theorem lies in not having to take the inverse of F(s) in order to find out the final value of f(t) in the time domain. This is particularly useful in circuits and systems.

The Laplace Transform

Example: Final Value Theorem:

Given:

$$F(s)=\frac{(s+2)^2-3^2}{\left[(s+2)^2+3^2\right]} \qquad note\ F^{-1}(s)=te^{-2t}\cos 3t$$

Find $f(\infty)$.

$$f(\infty)=\lim_{s\to 0} sF(s)=\lim_{s\to 0} s\frac{(s+2)^2-3^2}{\left[(s+2)^2+3^2\right]}=0$$