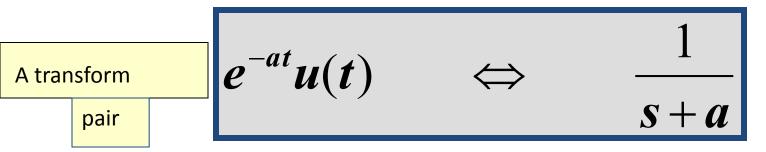
# Waveform synthesis and Laplace Transform of complex waveforms

Building transform pairs:

$$L[e^{-at}u(t)] = \int_{0}^{\infty} e^{-at}e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)} \Big|_{0}^{\infty} = \frac{1}{s+a}$$



Building transform pairs:

$$L[tu(t)] = \int_{0}^{\infty} te^{-st} dt$$

$$\int_{0}^{\infty} u dv = uv \Big|_{0}^{\infty} - \int_{0}^{\infty} v du$$

pair

$$tu(t) \iff \frac{1}{s^2}$$
 A transform pair

Building transform pairs:

$$L[\cos(wt)] = \int_{0}^{\infty} \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st} dt$$
$$= \frac{1}{2} \left[ \frac{1}{s - jw} - \frac{1}{s + jw} \right]$$
$$= \frac{s}{s^{2} + w^{2}}$$

$$\cos(wt)u(t) \iff \frac{s}{s^2 + w^2}$$
 A transform pair

**Time Shift** 

$$L[f(t-a)u(t-a)] = \int_{a}^{\infty} f(t-a)e^{-st}$$
  
Let  $x = t-a$ , then  $dx = dt$  and  $t = x+a$   
As  $t \to a$ ,  $x \to 0$  and as  $t \to \infty, x \to \infty$ . So,  
$$\int_{0}^{\infty} f(x)e^{-s(x+a)}dx = e^{-as}\int_{0}^{\infty} f(x)e^{-sx}dx$$

$$L[f(t-a)u(t-a)]=e^{-as}F(s)$$

Frequency Shift

$$L[e^{-at}f(t)] = \int_{0}^{\infty} [e^{-at}f(t)]e^{-st}dt$$
$$= \int_{0}^{\infty} f(t)e^{-(s+a)t}dt = F(s+a)$$

$$L[e^{-at}f(t)]=F(s+a)$$

Example: Using Frequency Shift

Find the L[e<sup>-at</sup>cos(wt)]

In this case, f(t) = cos(wt) so,

$$F(s) = \frac{s}{s^2 + w^2}$$
  
and 
$$F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

$$L[e^{-at}\cos(wt)] = \frac{(s+a)}{(s+a)^2 + (w)^2}$$

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Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

Integrate by parts:

$$u = e^{-st}, du = -se^{-st}dt \text{ and}$$
$$dv = \frac{df(t)}{dt} dt = df(t), so v = f(t)$$

\*note

Time Differentiation:

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_{0}^{\infty} - \int_{0}^{\infty} f(t)\left[-se^{-st}\right]dt$$
$$= 0 - f(0) + s\int_{0}^{\infty} f(t)e^{-st}dt$$

So we have shown:

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$