Waveform synthesis and Laplace Transform of complex waveforms

The Laplace Transform

Building transform pairs:

$$L[e^{-at}u(t)] = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty e} e^{-(s+a)t} dt$$

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)}\Big|_0^{\infty} = \frac{1}{s+a}$$

A transform pair

$$e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{s+a}$$

The Laplace Transform

Building transform pairs:

$$L[tu(t)] = \int_0^\infty te^{-st}dt$$

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

u = t
dv = e^{-st}dt

$$tu(t) \quad \Leftrightarrow \quad \frac{1}{s^2}$$

A transform pair

The Laplace Transform

Building transform pairs:

$$L[\cos(wt)] = \int_0^\infty \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st} dt$$

$$= \frac{1}{2}\left[\frac{1}{s - jw} - \frac{1}{s + jw}\right]$$

$$= \frac{s}{s^2 + w^2}$$

$$\cos(wt)u(t) \Leftrightarrow \frac{s}{s^2 + w^2}$$

A transform pair

The Laplace Transform

Time Shift

$$L[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-st}$$

$$Let\ x = t-a, then\ dx = dt\ and\ t = x+a$$

$$As\ t \to a,\ x \to 0\ and\ as\ t \to \infty, x \to \infty.\ So,$$

$$\int_0^\infty f(x)e^{-s(x+a)}dx = e^{-as}\int_0^\infty f(x)e^{-sx}dx$$

$$\boxed{L[f(t-a)u(t-a)] = e^{-as}F(s)}$$

The Laplace Transform

Frequency Shift

$$L[e^{-at}f(t)] = \int_0^\infty [e^{-at}f(t)]e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

$$L[e^{-at}f(t)] = F(s+a)$$

The Laplace Transform

Example: Using Frequency Shift

Find the L[e^{-at}cos(wt)]

In this case, f(t) = cos(wt) so,

$$F(s) = \frac{s}{s^2 + w^2}$$

$$and\ \ F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

$$L[e^{-at}\cos(wt)] = \frac{(s+a)}{(s+a)^2 + (w)^2}$$

The Laplace Transform

Example: Using Frequency Shift

Find the L[e^{-at}cos(wt)]

In this case, f(t) = cos(wt) so,

$$F(s) = \frac{s}{s^2 + w^2}$$

$$and \;\; F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

$$L[e^{-at}\cos(wt)] = \frac{(s+a)}{(s+a)^2 + (w)^2}$$

The Laplace Transform

Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

Integrate by parts:

$$u = e^{-st},\ du = -se^{-st}dt \ \ and$$

$$dv = \frac{df(t)}{dt}dt = df(t),\ so\ \ v = f(t)$$

*note

The Laplace Transform

Time Differentiation:

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty} f(t)\left[-se^{-st}\right]dt$$

$$= 0 - f(0) + s\int_0^{\infty} f(t)e^{-st}dt$$

So we have shown:

$$\boxed{L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)}$$