Introduction, State Space representation of linear systems

STATE VARIABLE MODELS

We consider physical sytems described by nth-order ordinary differential equation. Utilizing a set of variables, known as state variables, we can obtain a set of first-order differential equations. We group these first-order equations using a compact matrix notation in a model known as the state variable model.

The time-domain state variable model lends itself readily to computer solution and analysis. The Laplace transform is utilized to transform the differential equations representing the system to an algebraic equation expressed in terms of the complex variable ***s***. Utilizing this algebraic equation, we are able to obtain a transfer function representation of the input-output relationship.

With the ready availability of digital computers, it is convenient to consider the time-domain formulation of the equations representing control system. The time domain techniques can be utilized for nonlinear, time varying, and multivariable systems.

A time-varying control system is a system for which one or more of the parameters of the system may vary as a function of time.

For example, the mass of a missile varies as a function of time as the fuel is expended during flight. A multivariable system is a system with several input and output.

The State Variables of a Dynamic System:

The time-domain analysis and design of control systems utilizes the concept of the state of a system.

The state of a system is a set of variables such that the knowledge of these variables and the input functions will, with the equations describing the dynamics, provide the future state and output of the system.

For a dynamic system, the state of a system is described in terms of a set of state variables

$$[x_1(t) \quad x_2(t) \quad \ldots \quad x_n(t)]$$

The state variables are those variables that determine the future behavior of a system when the present state of the system and the excitation signals are known. Consider the system shown in Figure 1, where $y_1(t)$ and $y_2(t)$ are the output signals and $u_1(t)$ and $u_2(t)$ are the input signals. A set of state variables $[x_1\ x_2\ \ldots\ x_n]$ for the system shown in the figure is a set such that knowledge of the initial values of the state variables $[x_1(t_0)\ x_2(t_0)\ \ldots\ x_n(t_0)]$ at the initial time t_0, and of the input signals $u_1(t)$ and $u_2(t)$ for $t>=t_0$, suffices to determine the future values of the outputs and state variables.

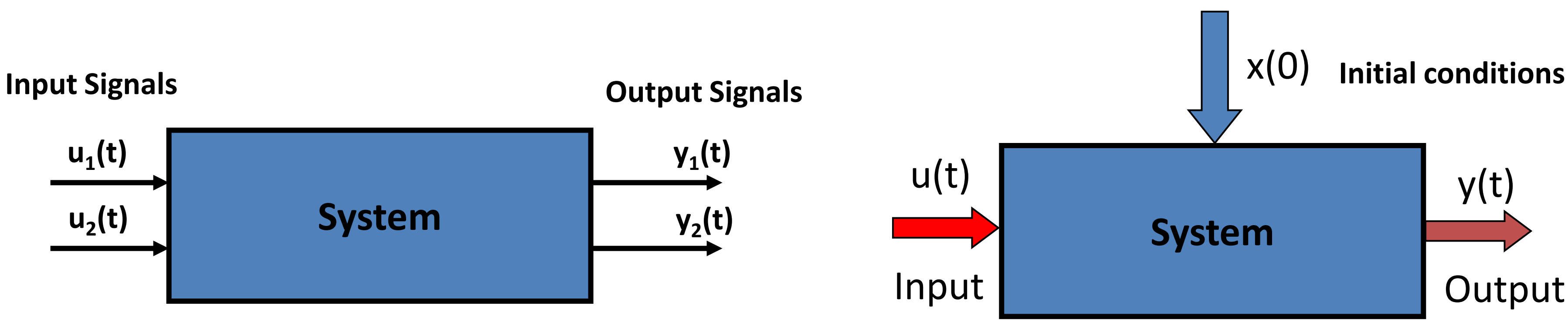

Figure 1. Dynamic system.

The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics.

A simple example of a state variable is the state of an on-off light switch. The switch can be in either the on or the off position, and thus the state of the switch can assume one of two possible values. Thus, if we know the present state (position) of the switch at t_0 and if an input is applied, we are able to determine the future value of the state of the element.

The concept of a set of state variables that represent a dynamic system can be illustrated in terms of the spring-mass-damper system shown in Figure 2. The number of state variables chosen to represent this system should be as small as possible in order to avoid redundant state variables. A set of state variables sufficient to describe this system includes the position and the velocity of the mass.

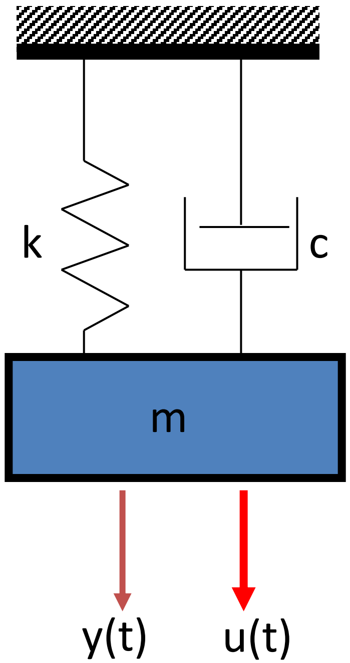

Figure 2. 1-dof system.

Therefore we will define a set of variables as [x_1 x_2], where

Kinetic and Potential energies, virtual work.

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt}$$

$$E_1 = \frac{1}{2} m \dot{y}^2, \quad E_2 = \frac{1}{2} k y^2, \quad \delta W = u(t)\delta y - c\dot{y}\delta y$$

Lagrangian of the system is expressed as $L = E_1 - E_2$

Generalized Force

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial(E_1 - E_2)}{\partial \dot{y}}\right) - \frac{\partial(E_1 - E_2)}{\partial y} = Q_y$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + kx = u(t)$$

$$m\frac{dx_2}{dt} + c\,x_2 + k\,x_1 = u(t)$$

Equation of motion in terms of state variables.

We can write the equations that describe the behavior of the spring-mass-damper system as the set of two first-order differential equations.

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u(t)$$

This set of difefrential equations describes the behavior of the state of the system in terms of the rate of change of each state variables.

As another example of the state variable characterization of a system, consider the RLC circuit shown in Figure 3.

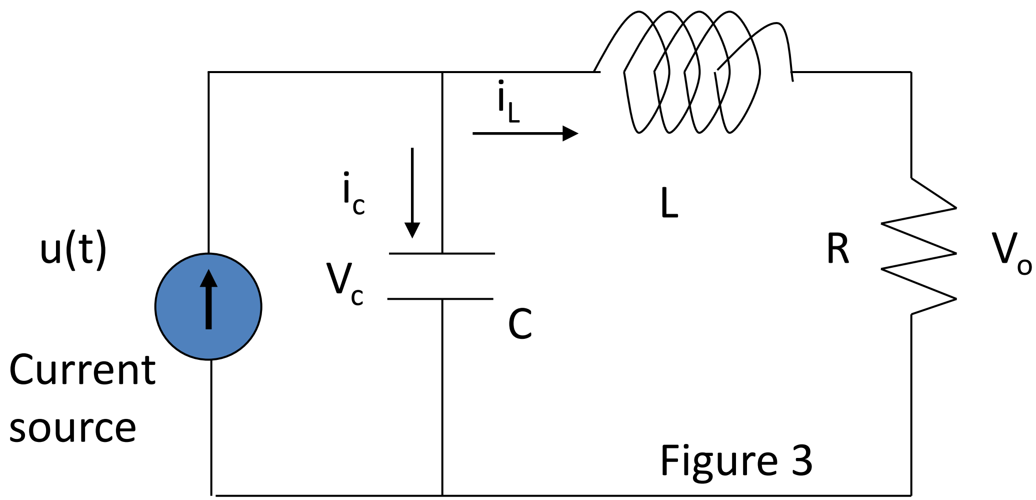

Figure 3

The state of this system can be described in terms of a set of variables [x_1 x_2], where x_1 is the capacitor voltage $v_c(t)$ and x_2 is equal to the inductor current $i_L(t)$. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables.

$$E_1 = \frac{1}{2}L\,i_L^{\;2}, \quad E_2 = \frac{1}{2C}\left(\int i_c dt\right)^2 = \frac{1}{2}C\,v_c^2$$

Therefore $x_1(t_0)$ and $x_2(t_0)$ represent the total initial energy of the network and thus the state of the system at $t=t_0$.

Utilizing Kirchhoff's current low at the junction, we obtain a first order differential equation by describing the rate of change of capacitor voltage

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L$$

Kirchhoff's voltage low for the right-hand loop provides the equation describing the rate of change of inducator current as

$$L\frac{di_L}{dt} = -R\,i_L + v_c$$

The output of the system is represented by the linear algebraic equation

$$v_0 = R\,i_L(t)$$

We can write the equations as a set of two first order differential equations in terms of the state variables x_1 [$v_C(t)$] and x_2 [$i_L(t)$] as follows:

$$C\frac{dv_c}{dt} = u(t) - i_L \quad \Longrightarrow \quad \frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$L\frac{di_L}{dt} = -R\,i_L + v_c \quad \Longrightarrow \quad \frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

The output signal is then $y_1(t) = v_0(t) = R\,x_2$

Utilizing the first-order differential equations and the initial conditions of the network represented by [$x_1(t_0)$ $x_2(t_0)$], we can determine the system's future and its output.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For the RLC circuit, we might choose the set of state variables as the two voltages, $v_C(t)$ and $v_L(t)$.