State Transition Matrix

In an actual system, there are several choices of a set of state variables that specify the *energy stored in a system* and therefore adequately describe the dynamics of the system.

The state variables of a system characterize the dynamic behavior of a system. The engineer’s interest is primarily in physical, where the variables are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.

The State Differential Equation:

The state of a system is described by the set of first-order differential equations written in terms of the state variables [x_1 x_2 ... x_n]. These first-order differential equations can be written in general form as

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + b_{11}u_1 + \cdots b_{1m}u_m$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n + b_{21}u_1 + \cdots b_{2m}u_m$$
$$\vdots$$
$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n + b_{n1}u_1 + \cdots b_{nm}u_m$$

Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

n: number of state variables, m: number of inputs.

The column matrix consisting of the state variables is called the **state vector** and is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Dorf and Bishop, Modern Control Systems

The vector of input signals is defined as u. Then the system can be represented by the compact notation of the state differential equation as

$$\dot{x} = Ax + Bu$$

This differential equation is also commonly called the state equation. The matrix **A** is an nxn square matrix, and **B** is an nxm matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation

$$y = Cx + Du$$

Where **y** is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.

We can write the state variable differential equation for the RLC circuit as

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

and the output as

$$y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax + bu$$

Where x(t) and u(t) are scalar functions of time. We expect an exponential solution of the form $e^{at.}$ Taking the Laplace transform of both sides, we have

$$s\,X(s) - x_0 = a\,X(s) + b\,U(s)$$

therefore,

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

The inverse Laplace transform of X(s) results in the solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}b\,u(\tau)\,d\tau$$

We expect the solution of the state differential equation to be similar to x(t) and to be of differential form. The **matrix exponential function** is defined as

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

which converges for all finite t and any A. Then the solution of the state differential equation is found to be

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

where we note that $[sI-A]^{-1}=\phi(s)$, which is the Laplace transform of $\phi(t)=e^{At}$. The matrix exponential function $\phi(t)$ describes the unforced response of the system and is called the fundamental or state transition matrix.

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$