Applications of State-Variable technique to the analysis of linear systems

CONTROLLABILITY:

Full-state feedback design commonly relies on **pole-placement techniques**. It is important to note that a system must be completely controllable and completely observable to allow the flexibility to place all the closed-loop system poles arbitrarily. The concepts of controllability and observability were introduced by Kalman in the 1960s.

A system is completely controllable if there exists an unconstrained control u(t) that can transfer any initial state $x(t_0)$ to any other desired location x(t) in a finite time, $t_0 \leq t \leq T$.

For the system

$$\dot{x} = Ax + Bu$$

we can determine whether the system is controllable by examining the algebraic condition

$$\text{rank}\left[B \quad AB \quad A^2B \cdots A^{n-1}B\right] = n$$

The matrix A is an nxn matrix an B is an nx1 matrix. For multi input systems, B can be nxm, where m is the number of inputs.

For a single-input, single-output system, the controllability matrix P_c is described in terms of A and B as

$$P_c = \left[B \quad AB \quad A^2B \cdots A^{n-1}B\right]$$

which is nxn matrix. Therefore, if the determinant of P_c is nonzero, the system is controllable.

Example:

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, AB = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix}, A^2B = \begin{bmatrix} 1 \\ -a_2 \\ \left(a_2^2 - a_1\right) \end{bmatrix}$$

$$P_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & \left(a_2^2 - a_1\right) \end{bmatrix}$$

The determinant of P_c =1 and ≠0 , hence this system is controllable.

Example.

Consider a system represented by the two state equations

$$\dot{x}_1 = -2x_1 + u \, , \; \dot{x}_2 = -3x_2 + d\,x_1$$

The output of the system is $y=x_2$. Determine the condition of controllability.

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } AB = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ d \end{bmatrix}$$

$$P_c = \begin{bmatrix} 1 & -2 \\ 0 & d \end{bmatrix}$$

The determinant of pc is equal to d, which is nonzero only when d is nonzero.

The controllability matrix P_c can be constructed in Matlab by using **ctrb** command.

From two-mass system,

$$B = \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -500 & 500 & -20.5 & 0 \\ 20000 & -20000 & 0 & -8.2 \end{bmatrix}$$

```
clc
clear
A=[0 0 1 0;0 0 0 1;-500 500 -20.5 0;20000 -
20000 0 -8.2];
B=[0;0;50;0];
Pc=ctrb(A,B)
rank_Pc=rank(Pc)
det_Pc=det(Pc)
```

```
Pc =

  1.0e+007 *

        0    0.0000   -0.0001   -0.0004
        0         0         0    0.1000
   0.0000   -0.0001   -0.0004    0.0594
        0         0    0.1000   -2.8700


rank_Pc =

     4


det_Pc =

 -2.5000e+015
```

The system is controllable.