Concept of Z-Transform

The z-Transform: Introduction

- Why z-Transform?
  1. Many of signals (such as $x(n)=u(n)$, $x(n) = (0.5)^n u(-n)$, $x(n) = \sin(n\omega)$ etc.) do not have a DTFT.
  2. Advantages like Fourier transform provided:
     - Solution process reduces to a simple algebraic procedures
     - The temporal domain sequence output $y(n) = x(n)*h(n)$ can be represent as $Y(z)= X(z)H(z)$
     - Properties of systems can easily be studied and characterized in z – domain (such as stability..)
- Topics:
  - Definition of z –Transform
  - Properties of z- Transform
  - Inverse z- Transform

Definition of the z-Transform

1. Definition:The z-transform of a discrete-time signal *x(n)* is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

   where $z = re^{jw}$ is a complex variable. The values of z for which the sum converges define a region in the z-plane referred to as the *region of convergence* (ROC).

2. Notationally, if *x(n)* has a z-transform *X(z),* we write

$$x(n) \overset{Z}{\longleftrightarrow} X(z)$$

3. The z-transform may be viewed as the DTFT or an exponentially weighted sequence. Specifically, note that with $z = re^{jw}$, $X(z)$ *can be looked as the DTFT of the sequence* $r^{-n}x(n)$ *and ROC is determined by the range of values of r of the following right inequation.*

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} [r^{-n}x(n)]e^{-jn\omega} \qquad \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty$$

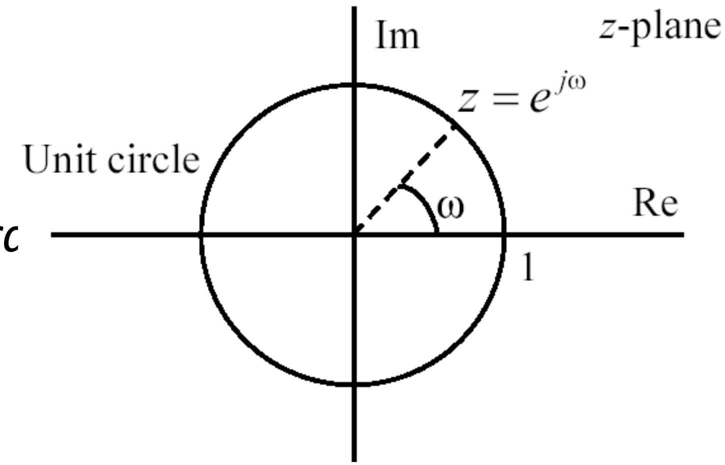
ROC & z-plane

- Complex z-plane

  $z = \mathcal{R}e(z) + jIm(z) = re^{jw}$

- *Zeros and poles of X(z)*

  Many signals have z-transforms that are rc
  function of z:



$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b(k) z^{-k}}{\sum_{k=0}^{p} a(k) z^{-k}}$$

  Factorizing it will give:

$$X(z) = C \frac{\prod_{k=1}^{q} (1 - \beta_k z^{-1})}{\prod_{k=1}^{p} (1 - \alpha_k z^{-1})}$$

  The roots of the numerator polynomial, β_k, are referred to as the zeros (o) and α_k are referred to as poles (x). ROC of X(z) will not contain poles.

ROC properties

- ROC is an annulus or disc in the z-plane centred at the origin. i.e. $0 \le r_R < |z| < r_L \le \infty.$
- A finite-length sequence has a z-transform with a region of convergence that includes the entire z-plane except, possibly, $z = 0$ and $z = \infty$. The point $z = \infty$ will be included if $x(n) = 0$ for *n* < 0, and the point $z = 0$ will be included if $x(n) = 0$ for $n > 0$.
- A right-sided sequence has a z-transform with a region of convergence that is the *exterior* of a circle:

  $$ROC:\ |z| > \alpha$$

- A left-sided sequence has a z-transform with a region of convergence that is the *interior* of a circle:

  $$ROC:\ |z| < \beta$$

- The Fourier Transform of x(n) converges absolutely if and only if ROC of z-transform includes the unit circle