Z-Transform of common functions

Example – Right-Sided Exponential Sequence (1)

- Consider $x[n]=a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence.

$$X(z)=\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require

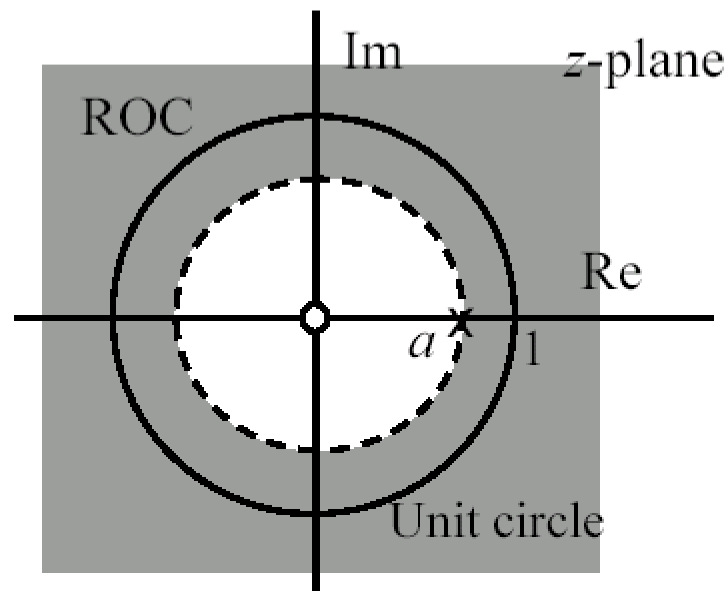
$$\sum_{n=-\infty}^{\infty} \left| az^{-1} \right|^n < \infty$$

Thus, the ROC is the range of values of z for which $|az^{-1}|<1$, or equivalently, $|z|>|a|$. Inside the ROC, the infinite series converges to

$$X(z)=\frac{1}{1-az^{-1}}=\frac{z}{z-a} \qquad |z|>|a|$$

Example – Right-Sided Exponential Sequence (2)

- The infinite sum becomes a simple rational function of z inside the ROC.
- Such a z-transform is determined to within a constant multiplier by its zeros and its poles.
- For this example,
  one zero: z=0 (plotted as o);
  one pole: z=a (plotted as x).
- When $|a|<1$, the ROC includes the unit circle.

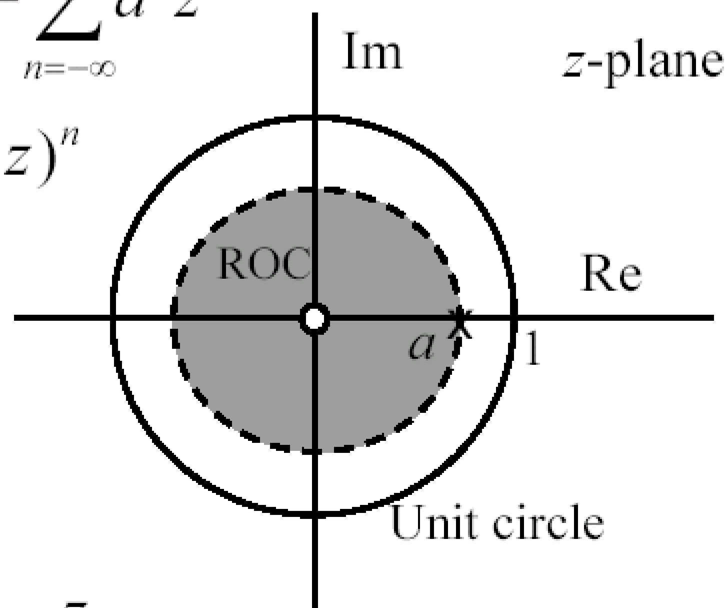


Example – Left-Sided Exponential Sequence

- Consider $x[n] = -a^n u[-n-1]$. Because it is nonzero only for $n \le -1$, this is an example of a *left-sided* sequence.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$



ROC

$$\left|a^{-1} z\right| < 1 \qquad |z| < |a|$$

and

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a} \qquad |z| < |a|$$

Notes on ROC

$$x[n] = a^n u[n] \quad \longleftrightarrow \quad X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > |a|$$

$$x[n] = -a^{-n} u[-n-1]$$

$$\longleftrightarrow \quad X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| < |a|$$

- As can be seen from the two examples, *the algebraic expression or pole-zero pattern does not completely specify the z-transform of a sequence; i.e., the ROC must also be specified.*