Inverse Z Transform & theorems

Example – Two-Sided Exponential Sequence

- Consider the sequence

$$x[n] = \underline{(-\tfrac{1}{3})^n u[n]} \; - \underline{(\tfrac{1}{2})^n u[-n-1]}$$

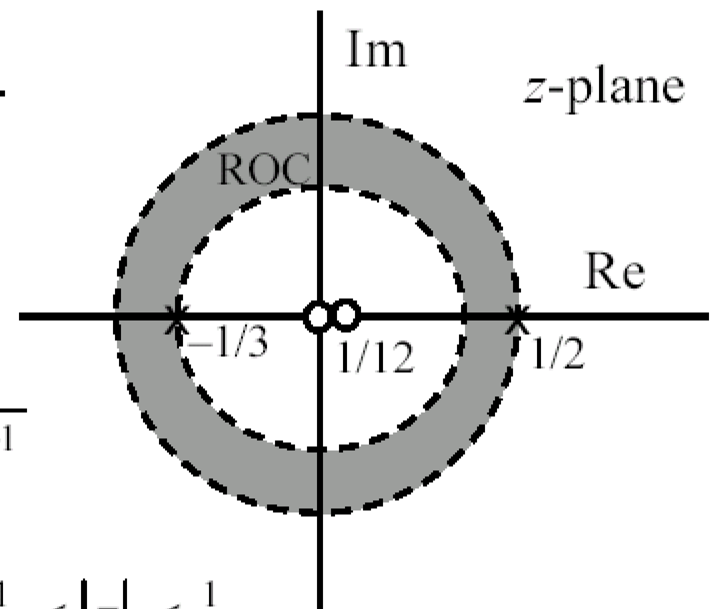
$$|z| > \tfrac{1}{3} \Updownarrow \qquad\qquad |z| < \tfrac{1}{2} \Updownarrow$$

$$\frac{1}{1+\frac{1}{3}z^{-1}} \qquad\qquad \frac{1}{1-\frac{1}{2}z^{-1}}$$

ROC $\quad \frac{1}{3} < |z| < \frac{1}{2}$

and $\quad X(z) = \dfrac{1}{1+\frac{1}{3}z^{-1}} + \dfrac{1}{1-\frac{1}{2}z^{-1}}$

$$= \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})} \qquad \frac{1}{3} < |z| < \frac{1}{2}$$



Example – Finite-Length Sequence

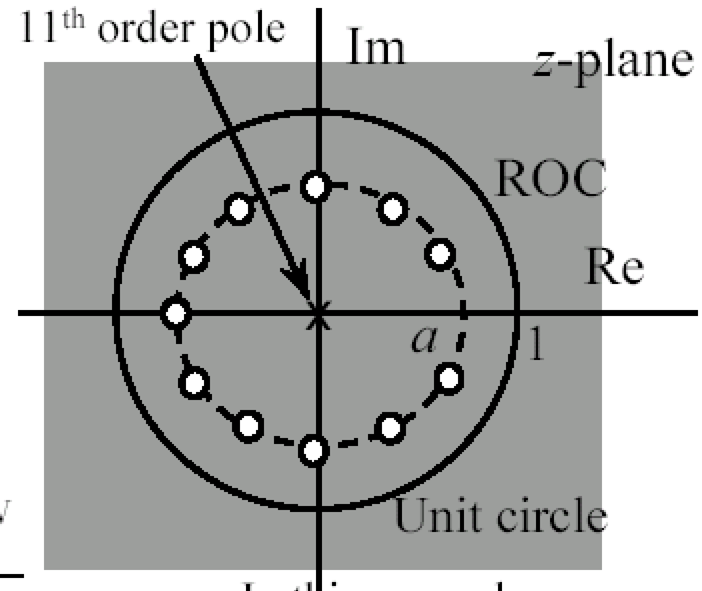
- Consider the sequence

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



In this example $N = 12$ and $0 < a < 1$.

The ROC is determined by $\sum_{n=0}^{N-1} |az^{-1}|^n$

which requires $|a| < \infty$ and $z \neq 0$

Table 4-1 Common z-Transform Pairs

Sequence	z-Transform	Region of Convergence
$\delta(n)$	1	all z
$\alpha^n u(n)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$-\alpha^n u(-n-1)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
$n\alpha^n u(n)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$-n\alpha^n u(-n-1)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$
$\cos(n\omega_0)u(n)$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$\sin(n\omega_0)u(n)$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$

Properties of Z-Transform

- Linearity

  If x(n) has a z-transform X(z) with a region of convergence Rx, and if y(n) has a z-transform Y(z) with a region of convergence Ry,

  $$w(n) = ax(n) + by(n) \overset{Z}{\longleftrightarrow} W(z) = aX(z) + bY(z)$$

  and the ROC of W(z) will include the intersection of Rx and Ry, that is, Rw contains $R_x \cap R_y$.

- Shifting property

  If x(n) has a z-transform X(z),

  $$x(n - n_0) \overset{Z}{\longleftrightarrow} z^{-n_0} X(z)$$

- Time reversal

  If x(n) has a z-transform X(z) with a region of convergence Rx that is the annulus $\alpha < |z| < \beta$, the z-transform of the time-reversed sequence x(-n) is

  $$x(-n) \overset{Z}{\longleftrightarrow} X(z^{-1})$$

  and has a region of convergence $1/\beta < |z| < 1/\alpha$, which is denoted by $1/R_x$

Properties of Z-Transform

- Multiplication by an exponential
  - If a sequence x(n) is multiplied by a complex exponential α^n.

$$\alpha^n x(n) \overset{Z}{\longleftrightarrow} X(\alpha^{-1} z)$$

- Convolution theorm

  If x(n) has a z-transform X(z) with a region of convergence R_x, and if h(n) has a z-transform H(z) with a region of convergence R_h,

$$y(n) = x(n) * h(n) \overset{Z}{\longleftrightarrow} Y(z) = X(z)H(z)$$

  The ROC of Y(z) will include the intersection of R_x and R_h, that is,

$$R_y \text{ contains } R_x \cap R_h \text{ .}$$

  With x(n), y(n), and h(n) denoting the input, output, and unit-sample response, respectively, and X(z), Y(x), and H(z) their z-transforms. The z-transform of the unit-sample response is often referred to as the system function.

- Conjugation

  If X(z) is the z-transform of x(n), the z-transform of the complex conjugate of x(n) is

$$x^*(n) \overset{Z}{\longleftrightarrow} X^*(z^*)$$

Properties of Z-Transform

- Derivative
  - If X(z) is the z-transform of x(n), the z-transform of is

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

- Initial value theorem

  If X(z) is the z-transform of x(n) and x(n) is equal to zero for n<0, the initial value, x(0), maybe be found from X(z) as follows:

$$x(0) = \lim_{z \to \infty} X(z)$$

Table 4-2 Properties of the z-Transform

Property	Sequence	z-Transform	Region of Convergence
Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
Shift	$x(n - n_0)$	$z^{-n_0}X(z)$	R_x
Time reversal	$x(-n)$	$X(z^{-1})$	$1/R_x$
Exponentiation	$\alpha^n x(n)$	$X(\alpha^{-1}z)$	$\|\alpha\|R_x$
Convolution	$x(n) * y(n)$	$X(z)Y(z)$	Contains $R_x \cap R_y$
Conjugation	$x^*(n)$	$X^*(z^*)$	R_x
Derivative	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	R_x

Note: Given the z-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence R_x and R_y, respectively, this table lists the z-transforms of sequences that are formed from $x(n)$ and $y(n)$.