2.8 Error Detection and Correction

- It is physically impossible for any data recording or transmission medium to be 100% perfect 100% of the time over its entire expected useful life.
- As more bits are packed onto a square centimeter of disk storage, as communications transmission speeds increase, the likelihood of error increases-- sometimes geometrically.
- Thus, error detection and correction is critical to accurate data transmission, storage and retrieval.

2.8 Error Detection and Correction

- Check digits, appended to the end of a long number can provide some protection against data input errors.
  - The last character of UPC barcodes and ISBNs are check digits.
- Longer data streams require more economical and sophisticated error detection mechanisms.
- Cyclic redundancy checking (CRC) codes provide error detection for large blocks of data.

2.8 Error Detection and Correction

- Checksums and CRCs are examples of *systematic error detection*.
- In *systematic error detection* a group of error control bits is appended to the end of the block of transmitted data.
  - This group of bits is called a *syndrome*.
- CRCs are polynomials over the modulo 2 arithmetic field.

The mathematical theory behind modulo 2 polynomials is beyond our scope. However, we can easily work with it without knowing its theoretical underpinnings.

2.8 Error Detection and Correction

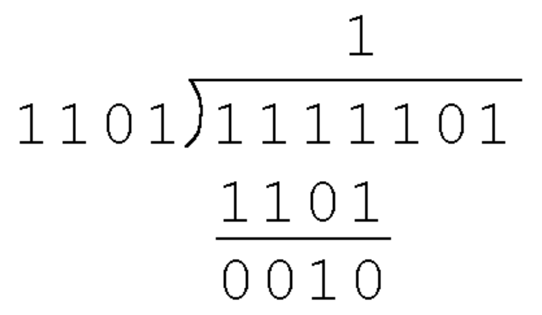
- Modulo 2 arithmetic works like clock arithmetic.
- In clock arithmetic, if we add 2 hours to 11:00, we get 1:00.
- In modulo 2 arithmetic if we add 1 to 1, we get 0. The addition rules couldn't be simpler:

$$
\begin{array}{ll}
0 + 0 = 0 & 0 + 1 = 1 \\
1 + 0 = 1 & 1 + 1 = 0
\end{array}
$$

You will fully understand why modulo 2 arithmetic is so handy after you study digital circuits in Chapter 3.

2.8 Error Detection and Correction

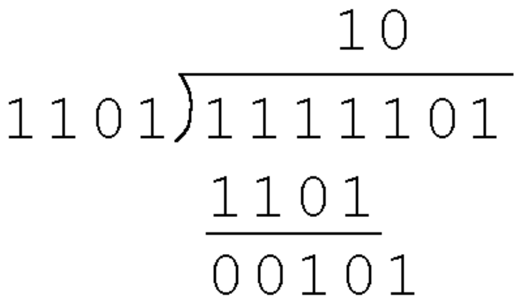
- **Find the quotient and remainder when 1111101 is divided by 1101 in modulo 2 arithmetic.**
  - As with traditional division, we note that the dividend is divisible once by the divisor.
  - We place the divisor under the dividend and perform modulo 2 subtraction.



$$\begin{array}{r} 1 \\ 1101\overline{)1111101} \\ \underline{1101} \\ 0010 \end{array}$$

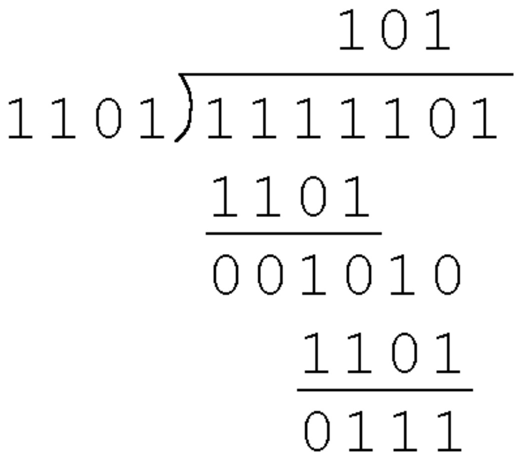
2.8 Error Detection and Correction

- **Find the quotient and remainder when 1111101 is divided by 1101 in modulo 2 arithmetic…**
  - Now we bring down the next bit of the dividend.
  - We see that 00101 is not divisible by 1101. So we place a zero in the quotient.



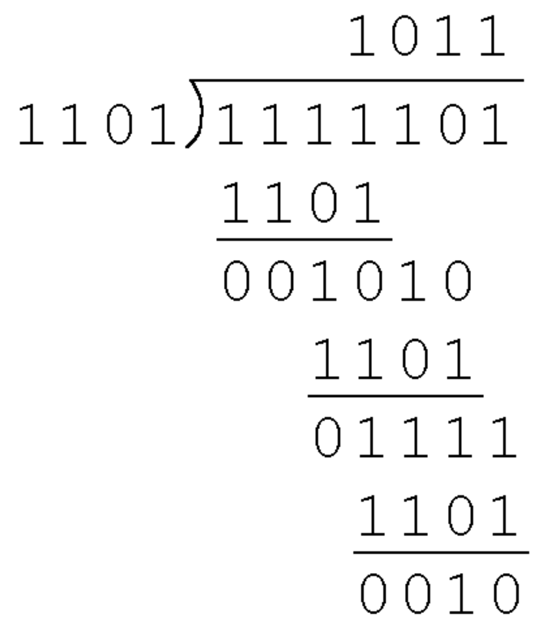
2.8 Error Detection and Correction

- **Find the quotient and remainder when 1111101 is divided by 1101 in modulo 2 arithmetic…**
  - 1010 is divisible by 1101 in modulo 2.
  - We perform the modulo 2 subtraction.



2.8 Error Detection and Correction

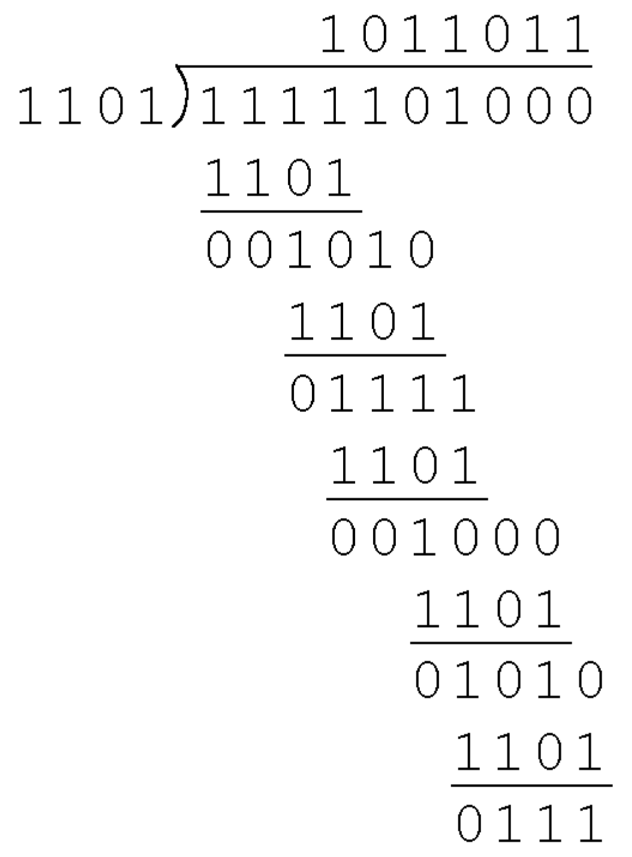
- **Find the quotient and remainder when 1111101 is divided by 1101 in modulo 2 arithmetic…**
  - We find the quotient is 1011, and the remainder is 0010.
- **This procedure is very useful to us in calculating CRC syndromes.**



Note: The divisor in this example corresponds to a modulo 2 polynomial: **$X^3 + X^2 + 1$.**

2.8 Error Detection and Correction

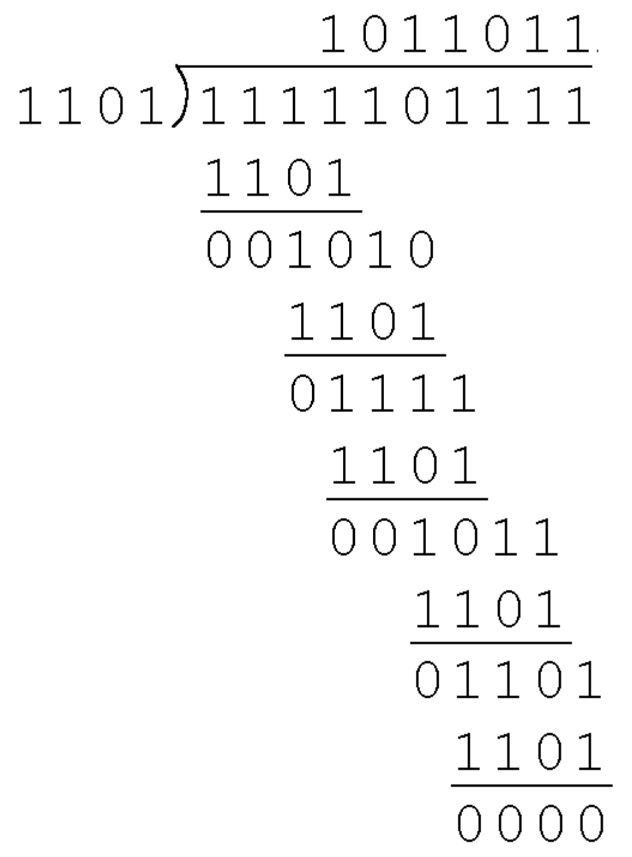
- **Suppose we want to transmit the information string: 1111101.**
- **The receiver and sender decide to use the (arbitrary) polynomial pattern, 1101.**
- **The information string is shifted left by one position less than the number of positions in the divisor.**
- **The remainder is found through modulo 2 division (at right) and added to the information string: 1111101000 + 111 = 1111101111.**



```
            1011011
     1101)1111101000
          1101
          001010
            1101
            01111
             1101
             001000
               1101
               01010
                1101
                0111
```

2.8 Error Detection and Correction

- **If no bits are lost or corrupted, dividing the received information string by the agreed upon pattern will give a remainder of zero.**
- **We see this is so in the calculation at the right.**
- **Real applications use longer polynomials to cover larger information strings.**
  - Some of the standard polynomials are listed in the text.



2.8 Error Detection and Correction

- Data transmission errors are easy to fix once an error is detected.
  - Just ask the sender to transmit the data again.
- In computer memory and data storage, however, this cannot be done.
  - Too often the only copy of something important is in memory or on disk.
- Thus, to provide data integrity over the long term, error *correcting* codes are required.

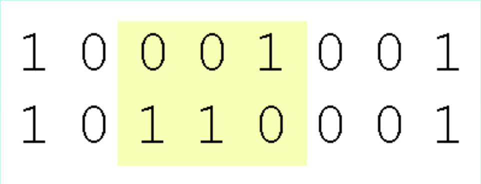
2.8 Error Detection and Correction

- Hamming codes and Reed-Soloman codes are two important error correcting codes.
- Reed-Soloman codes are particularly useful in correcting *burst errors* that occur when a series of adjacent bits are damaged.
  - Because CD-ROMs are easily scratched, they employ a type of Reed-Soloman error correction.
- Because the mathematics of Hamming codes is much simpler than Reed-Soloman, we discuss Hamming codes in detail.

2.8 Error Detection and Correction

- Hamming codes are code words formed by adding redundant check bits, or parity bits, to a data word.
- The *Hamming distance* between two code words is the number of bits in which two code words differ.

This pair of bytes has a
Hamming distance of 3:



1 0 0 0 1 0 0 1
1 0 1 1 0 0 0 1

- The minimum Hamming distance for a code is the smallest Hamming distance between *all* pairs of words in the code.

2.8 Error Detection and Correction

- The minimum Hamming distance for a code, D(min), determines its error detecting and error correcting capability.
- For any code word, *X,* to be interpreted as a different valid code word, *Y*, at least D(min) single-bit errors must occur in *X*.
- Thus, to detect *k* (or fewer) single-bit errors, the code must have a Hamming distance of D(min) = *k* + 1.

2.8 Error Detection and Correction

- Hamming codes can *detect* D(min) - 1 errors and *correct* $\left\lfloor \frac{D(Min) - 1}{2} \right\rfloor$ errors
- Thus, a Hamming distance of 2*k* + 1 is required to be able to correct *k* errors in any data word.
- Hamming distance is provided by adding a suitable number of parity bits to a data word.

2.8 Error Detection and Correction

- Suppose we have a set of *n*-bit code words consisting of *m* data bits and *r* (redundant) parity bits.
- An error could occur in any of the *n* bits, so each code word can be associated with *n* erroneous words at a Hamming distance of 1.
- Therefore,we have *n* + 1 bit patterns for each code word: one valid code word, and *n* erroneous words.

2.8 Error Detection and Correction

- With *n*-bit code words, we have 2^n possible code words consisting of 2^m data bits (where n = m + *r*).
- This gives us the inequality:

$$(n + 1) \times 2^m \leq 2^n$$

- Because n = m + *r*, we can rewrite the inequality as:

$$(m + r + 1) \times 2^m \leq 2^{m+r} \quad \text{or} \quad (m + r + 1) \leq 2^r$$

  - This inequality gives us a lower limit on the number of check bits that we need in our code words.

2.8 Error Detection and Correction

- Suppose we have data words of length $m = 4$. Then:

  $$(4 + r + 1) \leq 2^r$$

  implies that r must be greater than or equal to 3.
- This means to build a code with 4-bit data words that will correct single-bit errors, we must add 3 check bits.
- Finding the number of check bits is the hard part. The rest is easy.

2.8 Error Detection and Correction

- Suppose we have data words of length $m = 8$. Then:

  $$(8 + r + 1) \leq 2^r$$

  implies that r must be greater than or equal to 4.
- This means to build a code with 8-bit data words that will correct single-bit errors, we must add 4 check bits, creating code words of length 12.
- So how do we assign values to these check bits?

2.8 Error Detection and Correction

- With code words of length 12, we observe that each of the digits, 1 though 12, can be expressed in powers of 2. Thus:

$1 = 2^0$
$2 = 2^1$
$3 = 2^1 + 2^0$
$4 = 2^2$
$5 = 2^2 + 2^0$
$6 = 2^2 + 2^1$
$7 = 2^2 + 2^1 + 2^0$
$8 = 2^3$
$9 = 2^3 + 2^0$
$10 = 2^3 + 2^1$
$11 = 2^3 + 2^1 + 2^0$
$12 = 2^3 + 2^2$

  - 1 (= 2^0) contributes to all of the odd-numbered digits.
  - 2 (= 2^1) contributes to the digits, 2, 3, 6, 7, 10, and 11.
  - . . . And so forth . . .
- We can use this idea in the creation of our check bits.

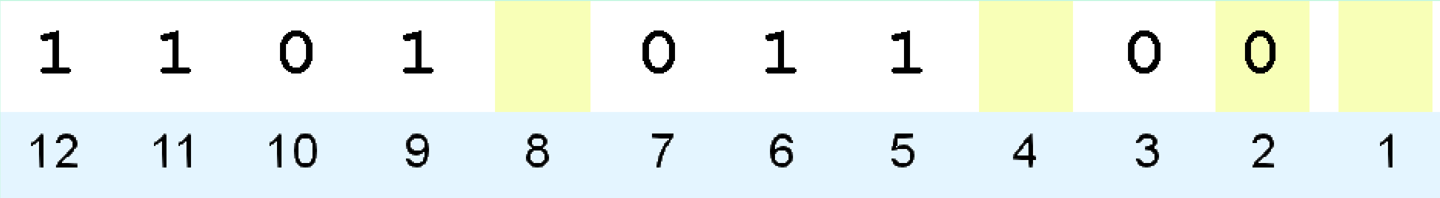
2.8 Error Detection and Correction

- Using our code words of length 12, number each bit position starting with 1 in the low-order bit.
- Each bit position corresponding to an even power of 2 will be occupied by a check bit.
- These check bits contain the parity of each bit position for which it participates in the sum.

_	_	_	_	P	_	_	_	P	_	P	P
12	11	10	9	8	7	6	5	4	3	2	1

2.8 Error Detection and Correction

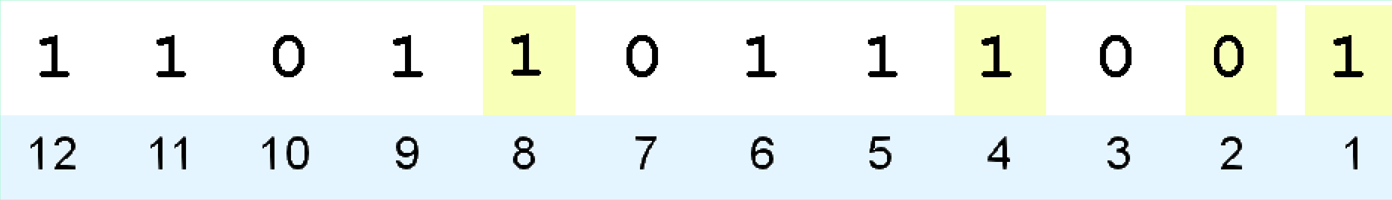
- Since 2 (= 2^1) contributes to the digits, 2, 3, 6, 7, 10, and 11. Position 2 will contain the parity for bits 3, 6, 7, 10, and 11.
- When we use even parity, this is the modulo 2 sum of the participating bit values.
- For the bit values shown, we have a parity value of 0 in the second bit position.



1	1	0	1		0	1	1		0	0	
12	11	10	9	8	7	6	5	4	3	2	1

What are the values for the other parity bits?

2.8 Error Detection and Correction



1	1	0	1	1	0	1	1	1	0	0	1
12	11	10	9	8	7	6	5	4	3	2	1

- The completed code word is shown above.
  - Bit 1checks the digits, 3, 5, 7, 9, and 11, so its value is 1.
  - Bit 4 checks the digits, 5, 6, 7, and 12, so its value is 1.
  - Bit 8 checks the digits, 9, 10, 11, and 12, so its value is also 1.
- Using the Hamming algorithm, we can not only detect single bit errors in this code word, but also correct them!

2.8 Error Detection and Correction

1	1	0	1	1	0	1	0	1	0	0	1
12	11	10	9	8	7	6	5	4	3	2	1

- Suppose an error occurs in bit 5, as shown above. Our parity bit values are:
  - Bit 1 checks digits, 3, 5, 7, 9, and 11. *Its value is 1, but should be zero.*
  - Bit 2 checks digits 2, 3, 6, 7, 10, and 11. The zero is correct.
  - Bit 4 checks digits, 5, 6, 7, and 12. *Its value is 1, but should be zero.*
  - Bit 8 checks digits, 9, 10, 11, and 12. This bit is correct.

2.8 Error Detection and Correction

1	1	0	1	1	0	1	0	1	0	0	1
12	11	10	9	8	7	6	5	4	3	2	1

- We have erroneous bits in positions 1 and 4.
- With *two* parity bits that don’t check, we know that the error is in the data, and not in a parity bit.
- Which data bits are in error? We find out by adding the bit positions of the erroneous bits.
- Simply, 1 + 4 = 5. This tells us that the error is in bit 5. If we change bit 5 to a 1, all parity bits check and our data is restored.