Error Analysis

- Two ways in which errors occur:
  - A is transmitted, AT+N<0 (0 received,1 sent)
  - -A is transmitted, -AT+N>0 (1 received,0 sent)

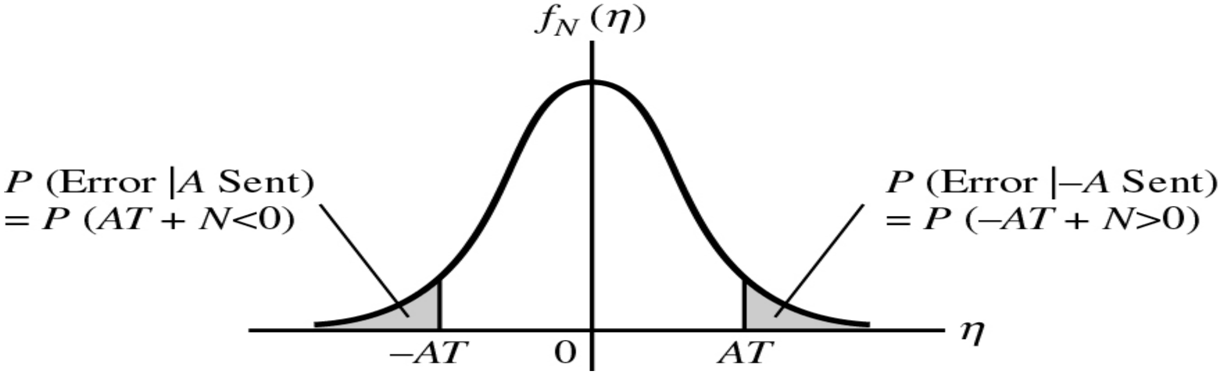


Figure 7-4 Illustration of error probabilities for binary signaling.

- $$P(Error \mid A) = \int_{-\infty}^{-AT} \frac{e^{-n^2/N_0T}}{\sqrt{\pi N_0 T}} dn = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

- Similarly,

$$P(Error \mid -A) = \int_{AT}^{\infty} \frac{e^{-n^2/N_0T}}{\sqrt{\pi N_0 T}} dn = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

- The average probability of error:

$$P_E = P(E \mid A)P(A) + P(E \mid -A)P(-A)$$

$$= Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

- Energy per bit:

$$E_b = \int_{t_0}^{t_0+T} A^2 dt = A^2 T$$

- Therefore, the error can be written in terms of the energy.
- Define

$$z = \frac{A^2 T}{N_0} = \frac{E_b}{N_0}$$

- Recall: Rectangular pulse of duration T seconds has magnitude spectrum

$$ATsinc(Tf)$$

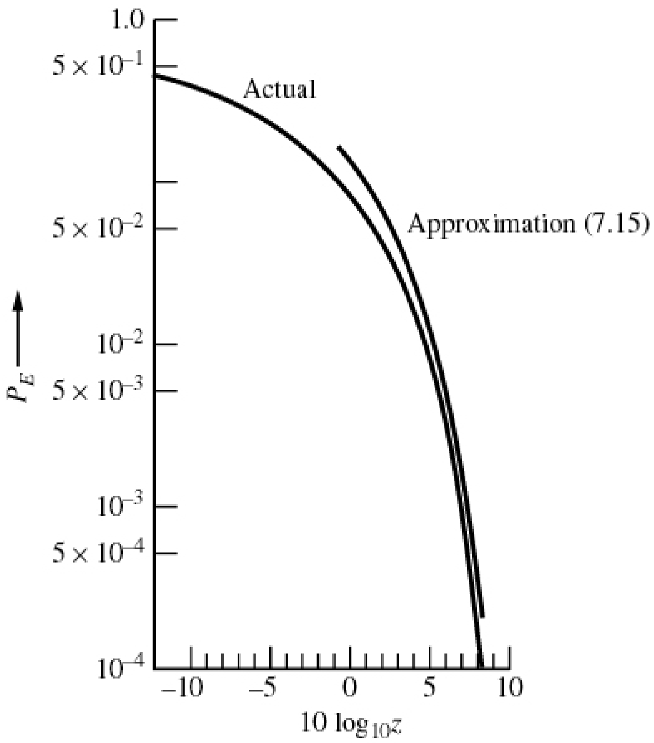
- Effective Bandwidth: $B_p = 1/T$
- Therefore,

$$z = \frac{A^2}{N_0 B_p}$$

- What's the physical meaning of this quantity?

Probability of Error vs. SNR

Figure 7-5
P_E for antipodal baseband digital signaling.



Error Approximation

- Use the approximation

$$Q(u) \cong \frac{e^{-u^2/2}}{u\sqrt{2\pi}}, u >> 1$$

$$P_E = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) \cong \frac{e^{-z}}{2\sqrt{\pi z}}, z >> 1$$

Example

- Digital data is transmitted through a baseband system with $N_0 = 10^{-7} W/Hz$, the received pulse amplitude A=20mV.

a)If 1 kbps is the transmission rate, what is probability of error?

$$B_p = \frac{1}{T} = \frac{1}{10^{-3}} = 10^3$$

$$SNR = z = \frac{A^2}{N_0 B_p} = \frac{400 \times 10^{-6}}{10^{-7} \times 10^3} = 400 \times 10^{-2} = 4$$

$$P_E \cong \frac{e^{-z}}{2\sqrt{\pi z}} = 2.58 \times 10^{-3}$$

b) If 10 kbps are transmitted, what must be the value of A to attain the same probability of error?

$$z = \frac{A^2}{N_0 B_p} = \frac{A^2}{10^{-7} \times 10^4} = 4 \Rightarrow A^2 = 4 \times 10^{-3} \Rightarrow A = 63.2 mV$$

- Conclusion:

Transmission power vs. Bit rate

Binary Signaling Techniques

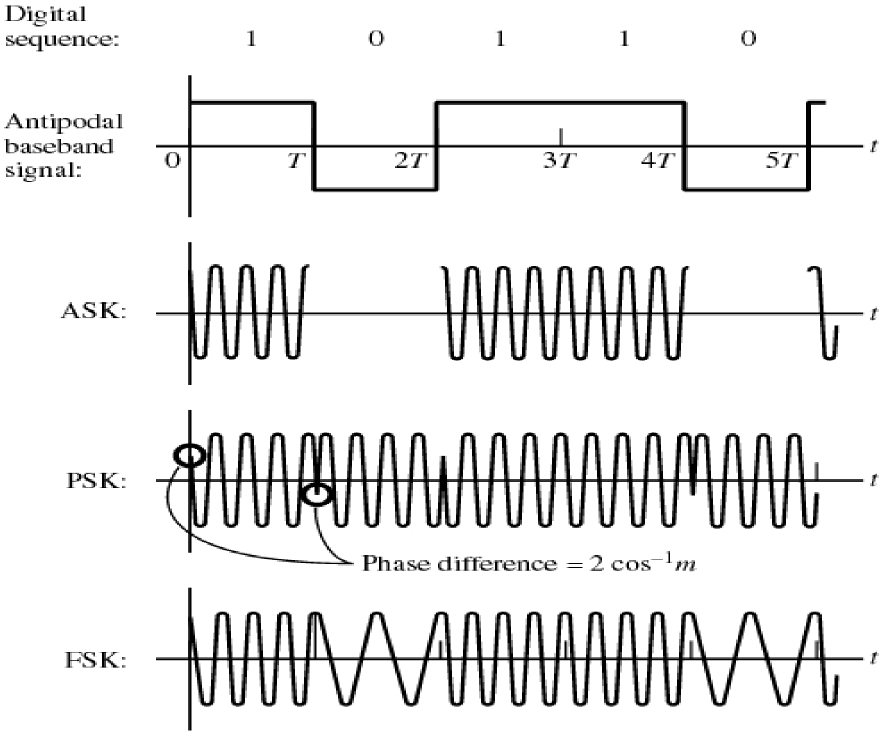


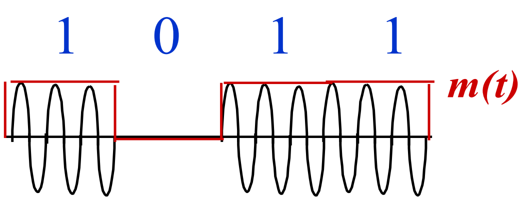
Figure 7-13
Waveforms for ASK, PSK, and FSK modulation.

Principles of Communications, 5/E by Rodger Ziemer and William Tranter

ASK, PSK, and FSK

- **Amplitude Shift Keying (ASK)**

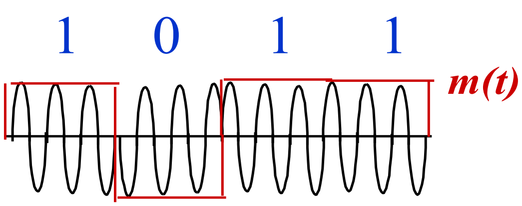
$$s(t) = m(t)A_c\cos(2\pi f_c t) = \begin{cases} A_c\cos(2\pi f_c t) & m(nT_b)=1 \\ 0 & m(nT_b)=0 \end{cases}$$



AM Modulation

- **Phase Shift Keying (PSK)**

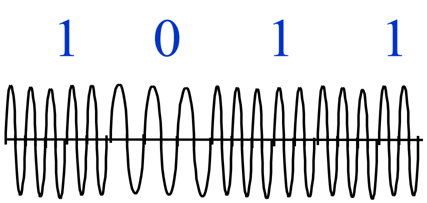
$$s(t) = A_c m(t)\cos(2\pi f_c t) = \begin{cases} A_c\cos(2\pi f_c t) & m(nT_b)=1 \\ A_c\cos(2\pi f_c t+\pi) & m(nT_b)=-1 \end{cases}$$



PM Modulation

- **Frequency Shift Keying**

$$s(t) = \begin{cases} A_c\cos(2\pi f_1 t) & m(nT_b)=1 \\ A_c\cos(2\pi f_2 t) & m(nT_b)=-1 \end{cases}$$



FM Modulation

Amplitude Shift Keying (ASK)

- 0→0
- 1→Acos(wct)
- What is the structure of the optimum receiver?

Receiver for binary signals in noise

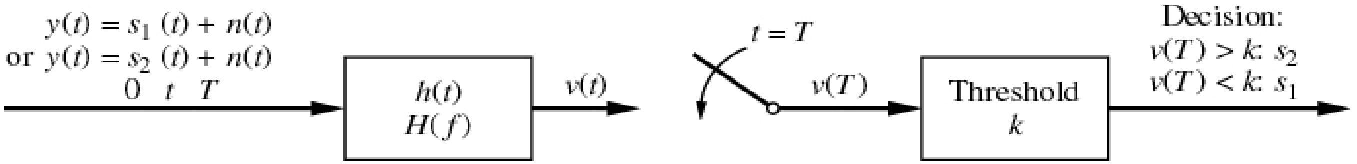


Figure 7-6 A possible receiver structure for detecting binary signals in white Gaussian noise.

Error Analysis

- 0 → s1(t), 1 → s2(t) in general.
- The received signal:

$$y(t) = s_1(t) + n(t), t_0 \le t \le t_0 + T$$

$$OR$$

$$y(t) = s_2(t) + n(t), t_0 \le t \le t_0 + T$$

- Noise is white and Gaussian.
- Find P_E
- In how many different ways can an error occur?

Error Analysis (general case)

- Two ways for error:
  - » Receive 1 → Send 0
  - » Receive 0 → Send 1
- Decision:
  - » The received signal is filtered. (How does this compare to baseband transmission?)
  - » Filter output is sampled every T seconds
  - » Threshold k
  - » Error occurs when:

$$v(T) = s_{01}(T) + n_0(T) > k$$

$$OR$$

$$v(T) = s_{02}(T) + n_0(T) < k$$

- s_{01}, s_{02}, n_0 are filtered signal and noise terms.
- Noise term: $n_0(t)$ is the filtered white Gaussian noise.
- Therefore, it's Gaussian (why?)
- Has PSD:

$$S_{n_0}(f) = \frac{N_0}{2} |H(f)|^2$$

- Mean zero, variance?
- Recall: Variance is equal to average power of the noise process

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{N_0}{2} |H(f)|^2 \, df$$

- The pdf of noise term is:

$$f_N(n) = \frac{e^{-n^2/2\sigma^2{}_0}}{\sqrt{2\pi\sigma^2}}$$

- Note that we still don't know what the filter is.
- Will any filter work? Or is there an optimal one?
- Recall that in baseband case (no modulation), we had the integrator which is equivalent to filtering with

$$H(f) = \frac{1}{j2\pi f}$$

- The input to the thresholder is:

$$V = v(T) = s_{01}(T) + N$$
$$OR$$
$$V = v(T) = s_{02}(T) + N$$

- These are also Gaussian random variables; why?
- Mean: $s_{01}(T) \quad OR \quad s_{02}(T)$
- Variance: Same as the variance of N

Distribution of V

- The distribution of V, the input to the threshold device is:

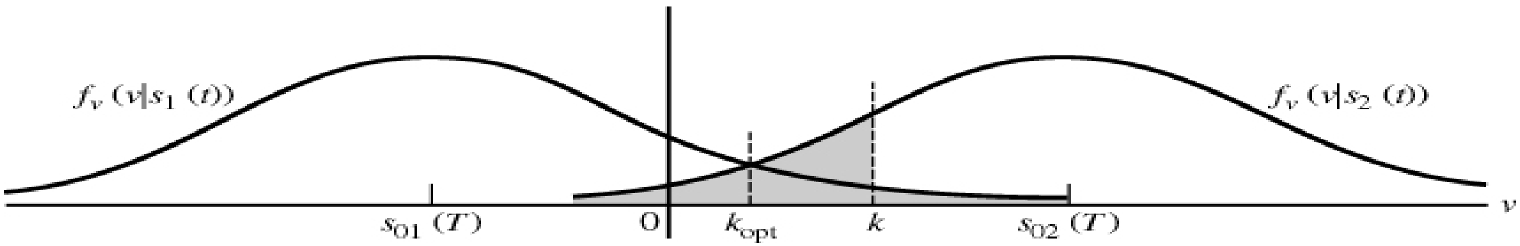


Figure 7-7 Conditional probability density functions of the filter output at time $t = T$.

Principles of Communications, 5/E by Rodger Ziemer and William Tranter

Probability of Error

- Two types of errors:

$$P(E \mid s_1(t)) = \int_k^{\infty} \frac{e^{-[v - s_{01}(T)]^2 / 2\sigma^2}}{\sqrt{2\pi\sigma^2}} dv = Q\left(\frac{k - s_{01}(T)}{\sigma}\right)$$

$$P(E \mid s_2(t)) = \int_{-\infty}^{k} \frac{e^{-[v - s_{02}(T)]^2 / 2\sigma^2}}{\sqrt{2\pi\sigma^2}} dv = 1 - Q\left(\frac{k - s_{02}(T)}{\sigma}\right)$$

- The average probability of error:

$$P_E = \frac{1}{2} P[E \mid s_1(t)] + \frac{1}{2} P[E \mid s_2(t)]$$

- Goal: Minimize the average probability of errror
- Choose the optimal threshold
- What should the optimal threshold, k_{opt} be?
- $K_{opt}=0.5[s_{01}(T)+s_{02}(T)]$
- $$P_E = Q\left(\frac{s_{02}(T)-s_{01}(T)}{2\sigma}\right)$$

Observations

- P_E is a function of the difference between the two signals.
- Recall: Q-function decreases with increasing argument. (Why?)
- Therefore, P_E will decrease with increasing distance between the two output signals
- Should choose the filter h(t) such that P_E is a minimum→maximize the difference between the two signals at the output of the filter

Matched Filter

- Goal: Given $s_1(t), s_2(t)$, choose H(f) such that $d = \frac{s_{02}(T) - s_{01}(T)}{\sigma}$ is maximized.
- The solution to this problem is known as the matched filter and is given by:

$$h_0(t) = s_2(T-t) - s_1(T-t)$$

- Therefore, the optimum filter depends on the input signals.

Matched filter receiver

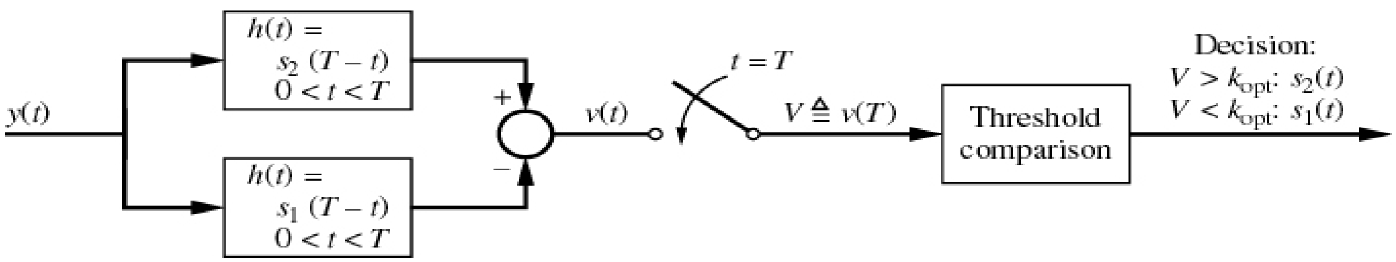


Figure 7-9 Matched filter receiver for binary signaling in white Gaussian noise.

Error Probability for Matched Filter Receiver

- Recall $P_E = Q\left(\frac{d}{2}\right)$
- The maximum value of the distance,

$$d_{\max}{}^2 = \frac{2}{N_0}(E_1 + E_2 - 2\sqrt{E_1 E_2}\,\rho_{12})$$

- E_1 is the energy of the first signal.
- E_2 is the energy of the second signal.

$$E_1 = \int_{t_0}^{t_0+T} s_1{}^2(t)\,dt$$

$$E_2 = \int_{t_0}^{t_0+T} s_2{}^2(t)\,dt$$

$$\rho_{12} = \frac{1}{\sqrt{E_1 E_2}} \int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt$$

- Therefore,

$$P_E = Q\left[\left(\frac{E_1 + E_2 - 2\sqrt{E_1 E_2}\,\rho_{12}}{2N_0}\right)^{1/2}\right]$$

- Probability of error depends on the signal energies (just as in baseband case), noise power, and the similarity between the signals.
- If we make the transmitted signals as dissimilar as possible, then the probability of error will decrease ($\rho_{12} = -1$)

ASK

$$s_1(t) = 0, s_2(t) = A\cos(2\pi f_c t)$$

- The matched filter: $A\cos(2\pi f_c t)$
- Optimum Threshold: $\frac{1}{4}A^2T$
- Similarity between signals?
- Therefore, $P_E = Q\left(\sqrt{\frac{A^2T}{4N_0}}\right) = Q\left(\sqrt{z}\right)$

- 3dB worse than baseband.

PSK

$$s_1(t) = A\sin(2\pi f_c t + \cos^{-1} m), s_2(t) = A\sin(2\pi f_c t - \cos^{-1} m)$$

- Modulation index: m (determines the phase jump)
- Matched Filter: $-2A\sqrt{1-m^2}\cos(2\pi f_c t)$
- Threshold: 0
- Therefore, $P_E = Q(\sqrt{2(1-m^2)z})$
- For m=0, 3dB better than ASK.

Matched Filter for PSK

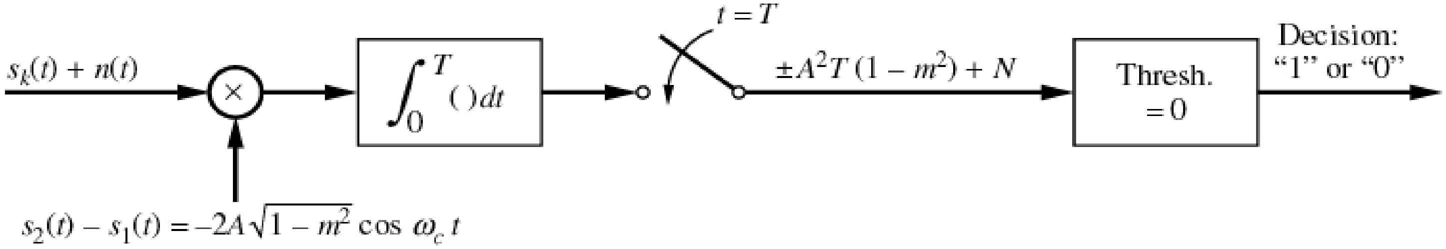


Figure 7-14 Correlator realization of optimum receiver for PSK.

FSK

- $s_1(t) = A\cos(2\pi f_c t), s_2(t) = A\cos(2\pi(f_c + \Delta f)t)$
- $\Delta f = \frac{m}{T}$
- Probability of Error: $Q(\sqrt{z})$
- Same as ASK

Applications

- Modems: FSK
- RF based security and access control systems
- Cellular phones