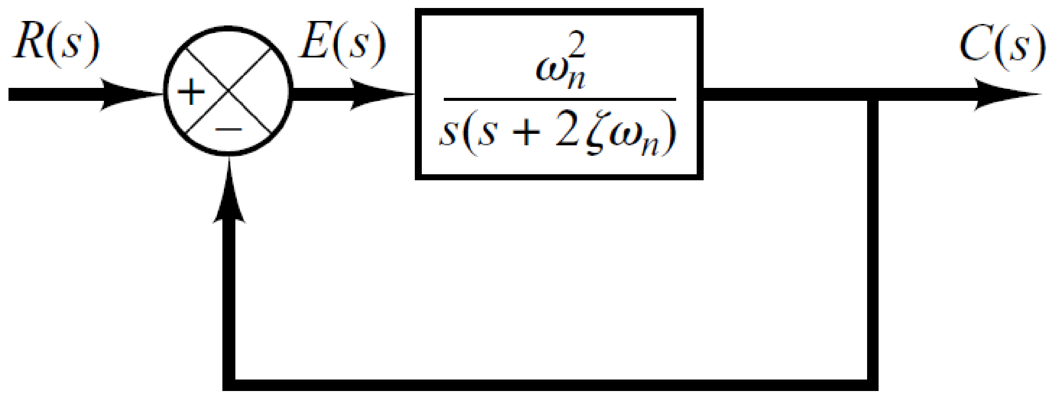
Time Response Specifications

Introduction

- A general second-order system is characterized by the following transfer function.



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n $\longrightarrow$ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

ζ $\longrightarrow$ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

Example#1

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \quad \Rightarrow \omega_n = 2\ rad/\sec$$

$$\cancel{s^2} + 2\zeta\omega_n s + \cancel{\omega_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

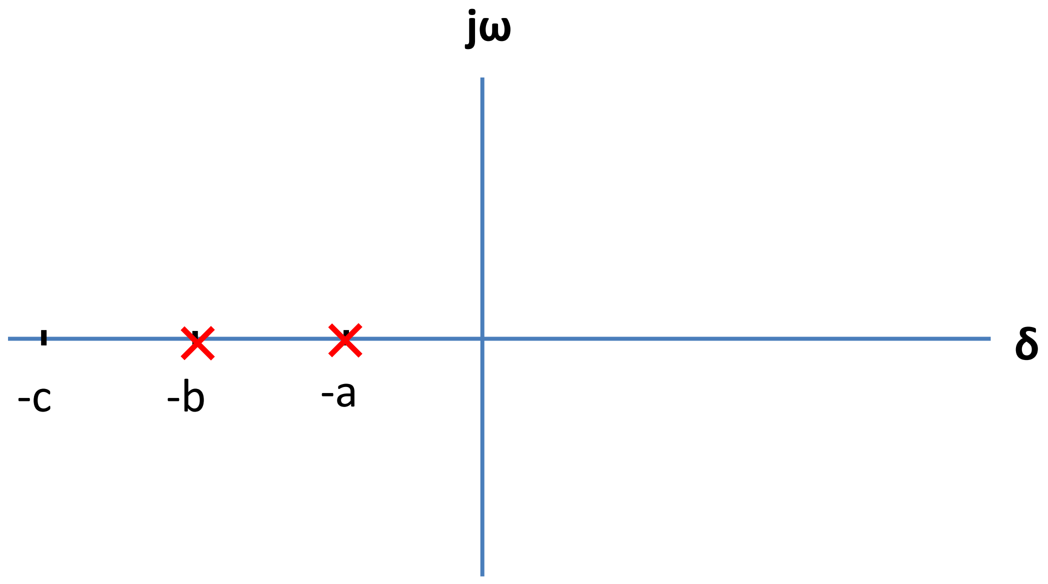
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ, a second-order system can be set into one of the four categories:

  1. *Overdamped* - when the system has two real distinct poles (ζ >1).

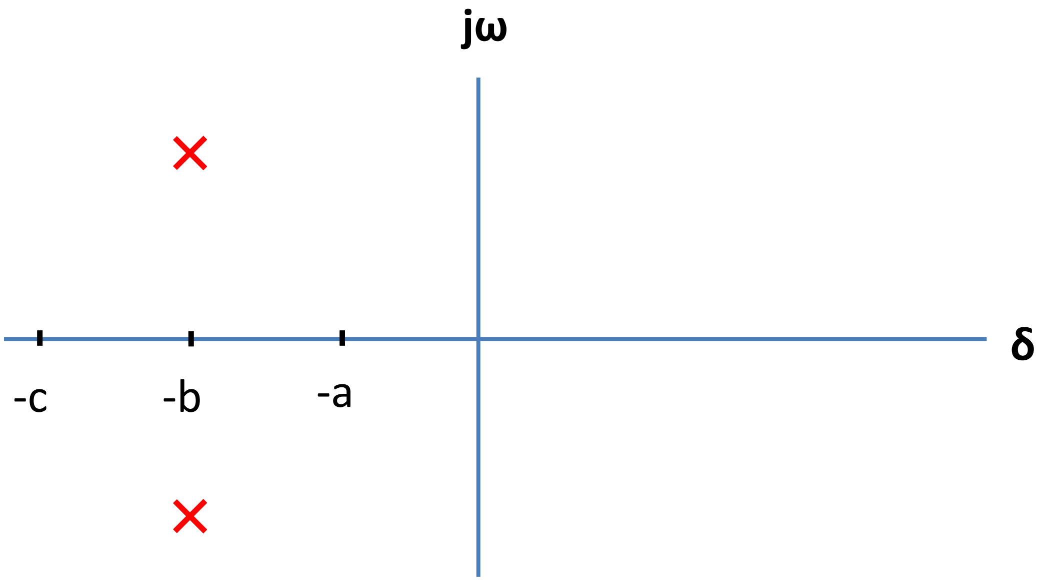


Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)



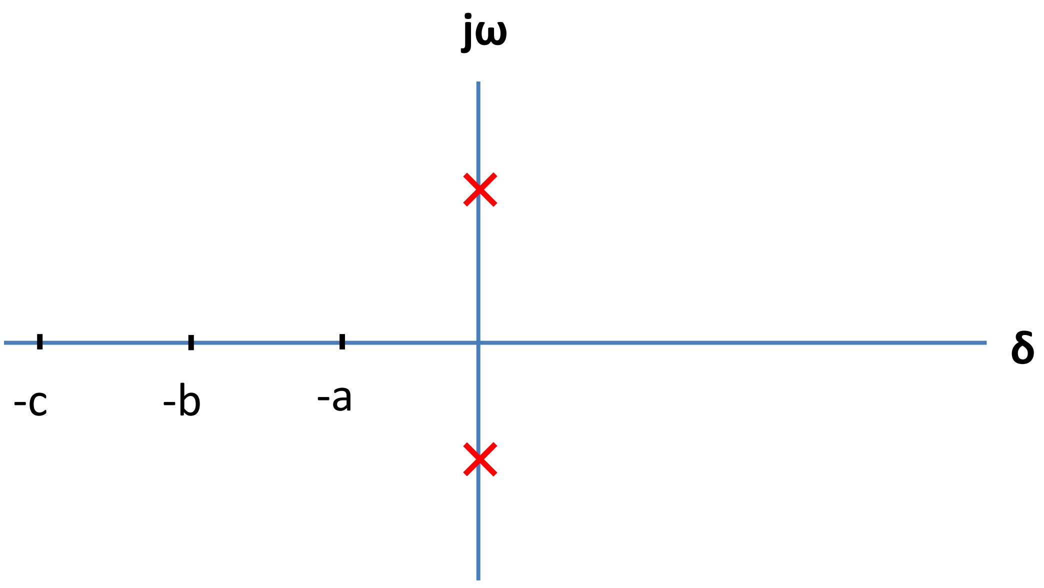
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

3. *Undamped* - when the system has two imaginary poles (ζ = 0).



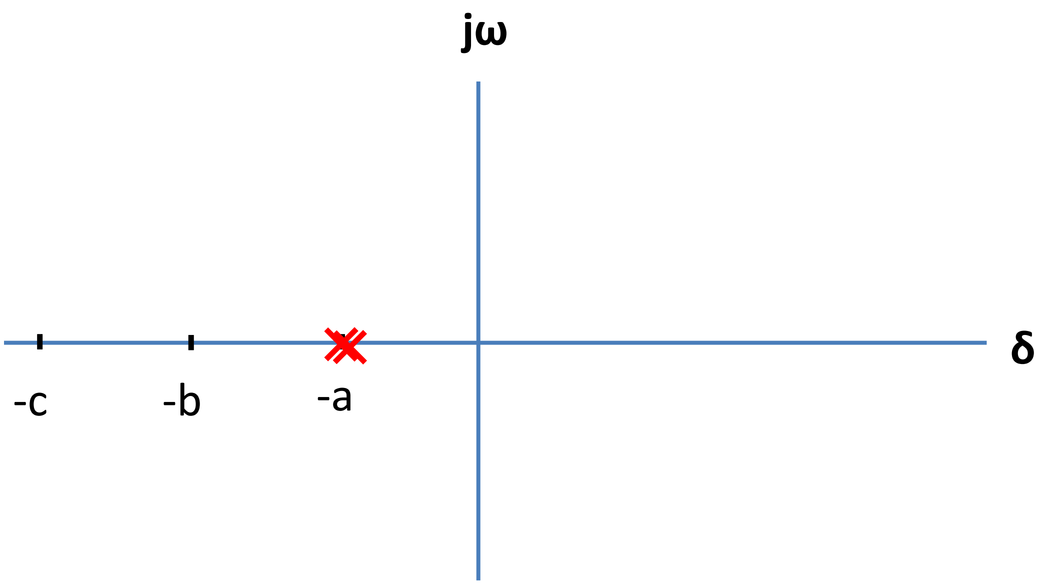
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

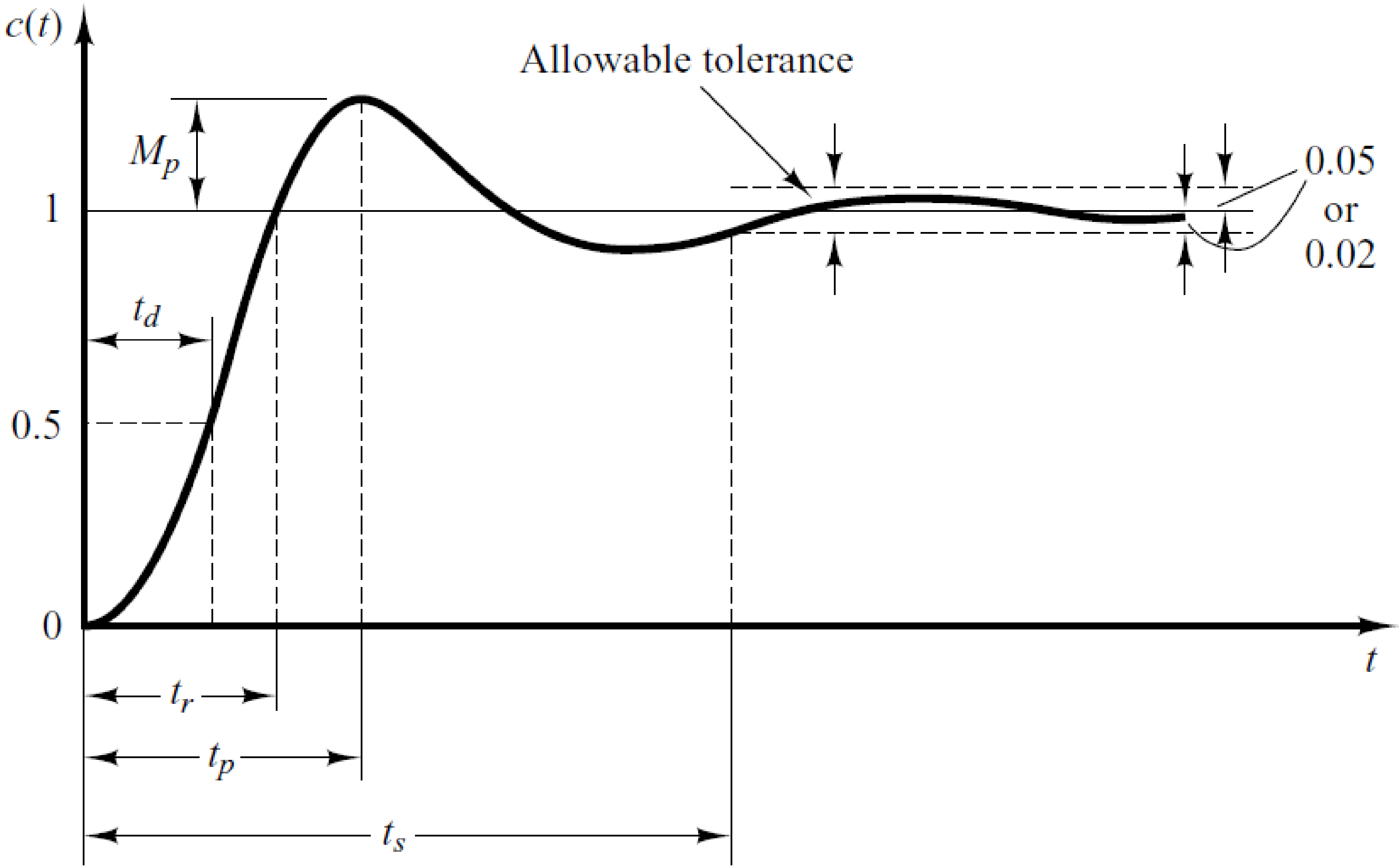
- According the value of ζ, a second-order system can be set into one of the four categories:

4. *Critically damped* - when the system has two real but equal poles (ζ = 1).



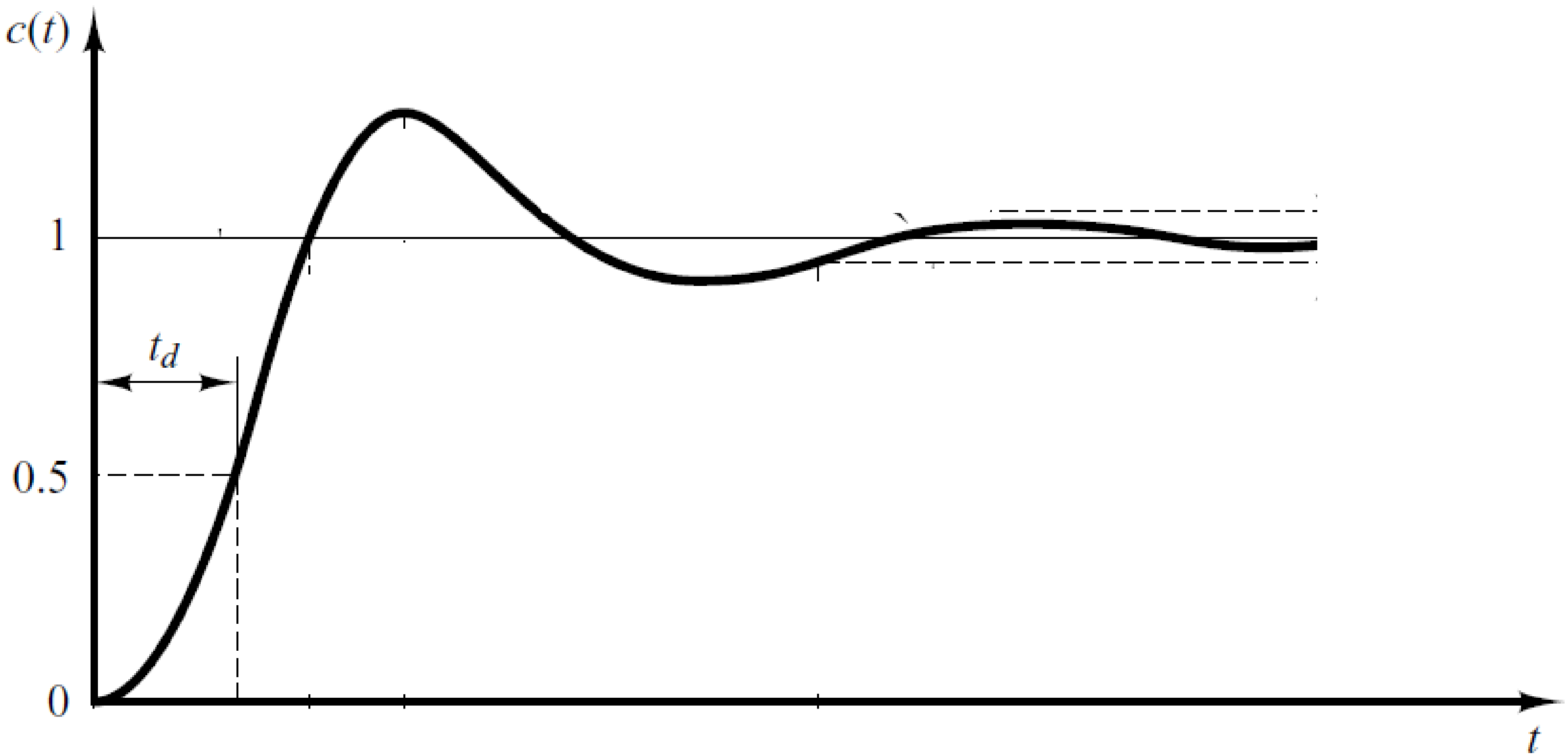
Time-Domain Specification

For $0<\zeta<1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input looks like



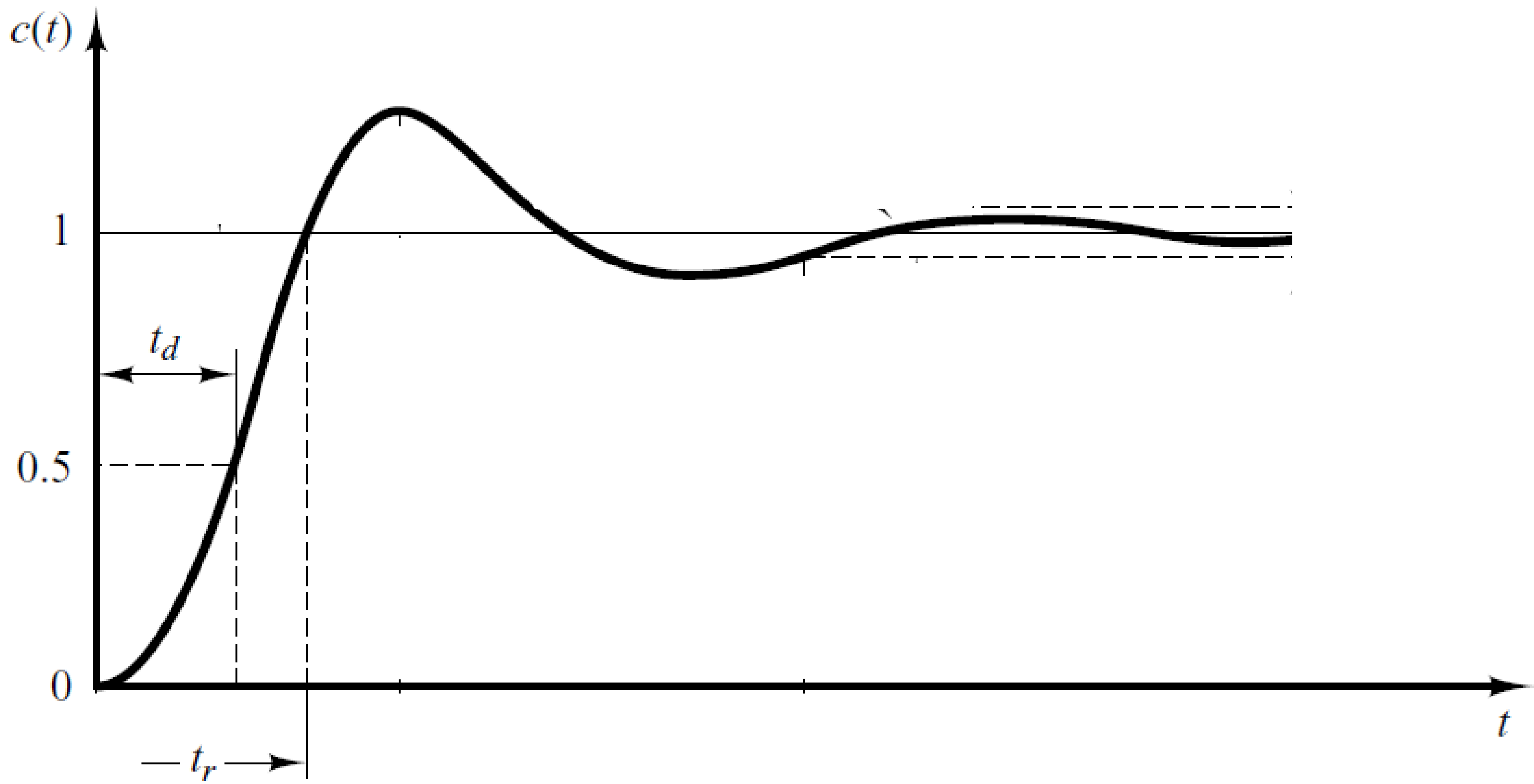
Time-Domain Specification

- The delay (t_d) time is the time required for the response to reach half the final value the very first time.



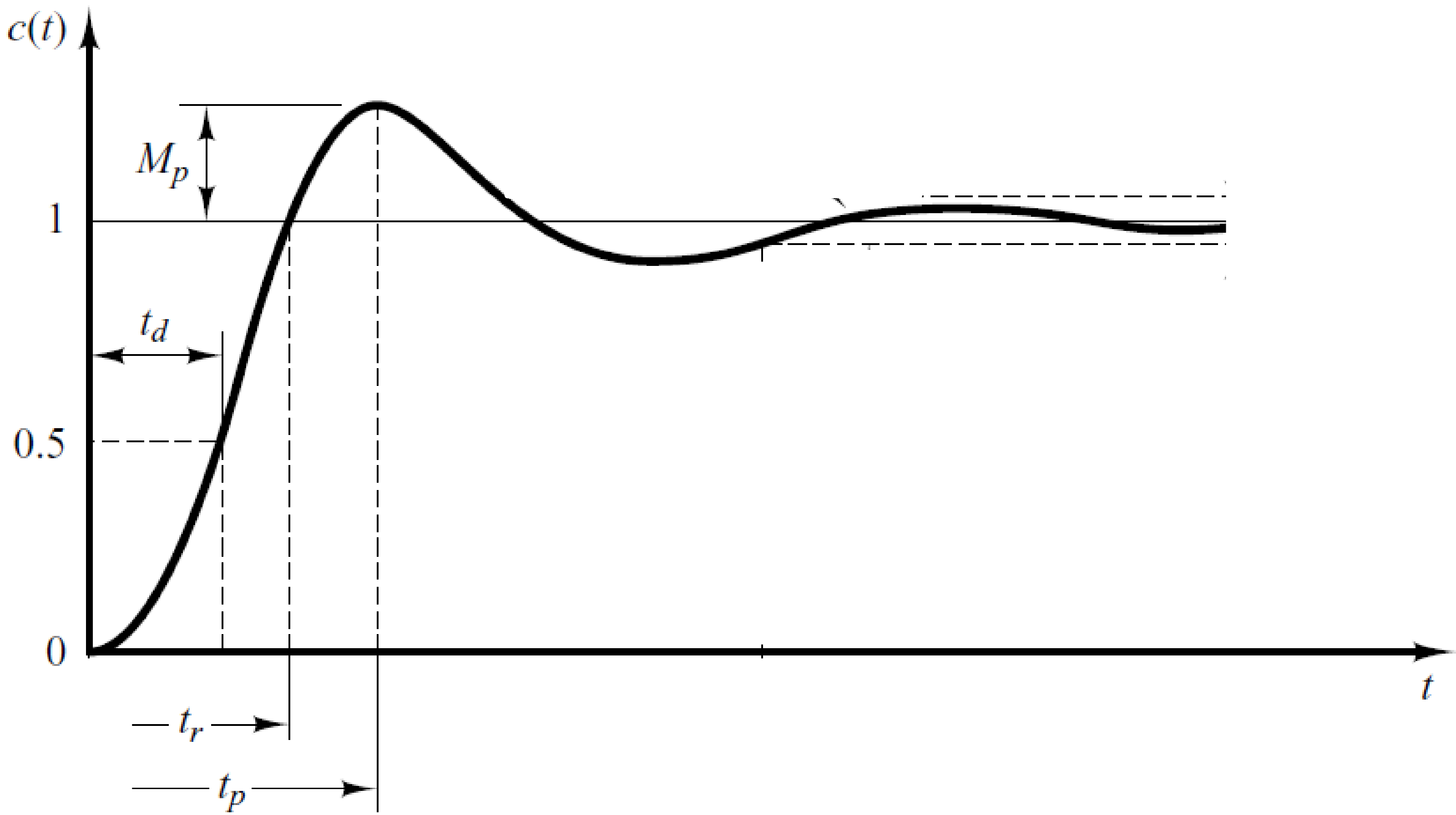
Time-Domain Specification

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



Time-Domain Specification

- The peak time is the time required for the response to reach the first peak of the overshoot.



Time-Domain Specification

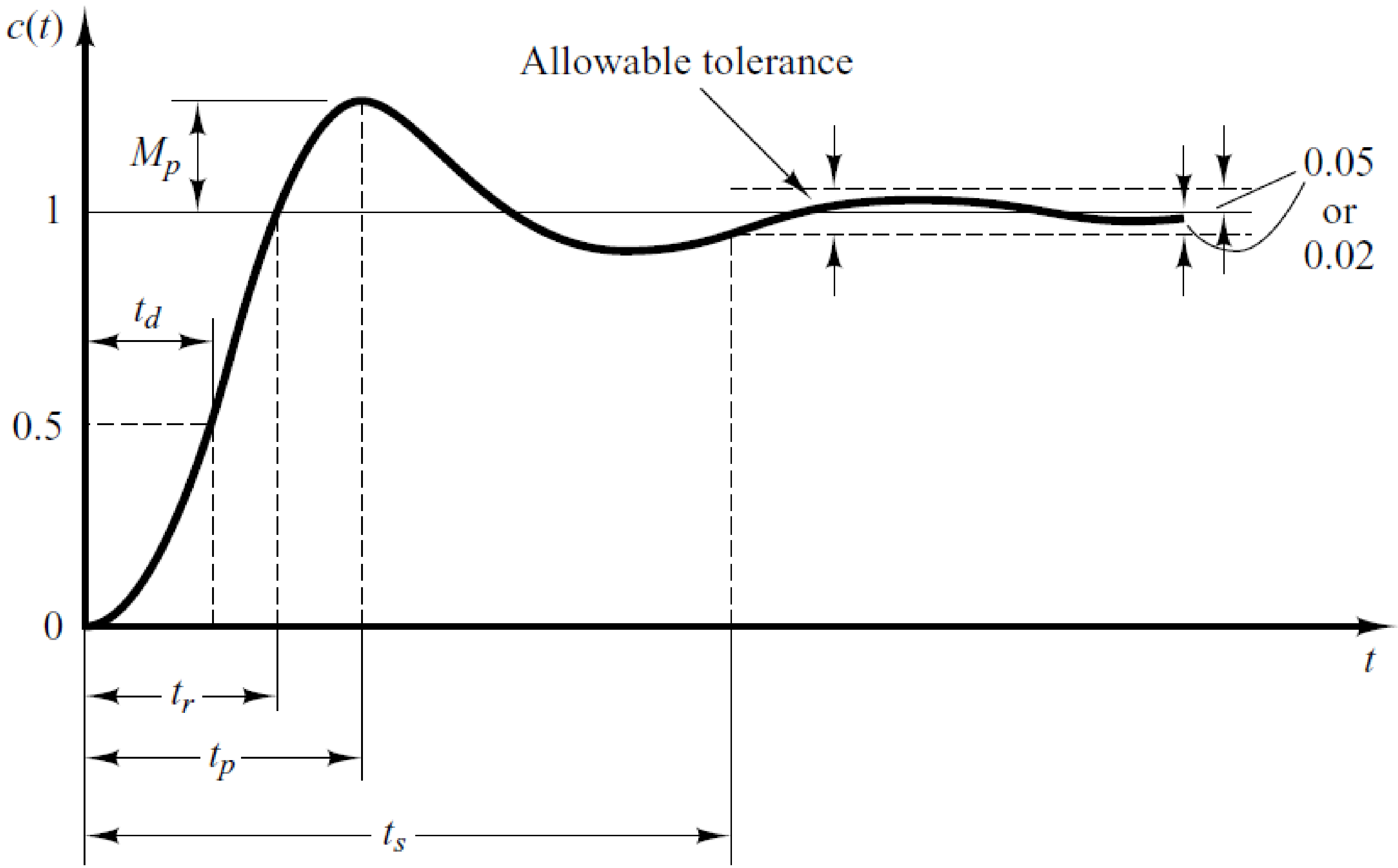
The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Time-Domain Specification

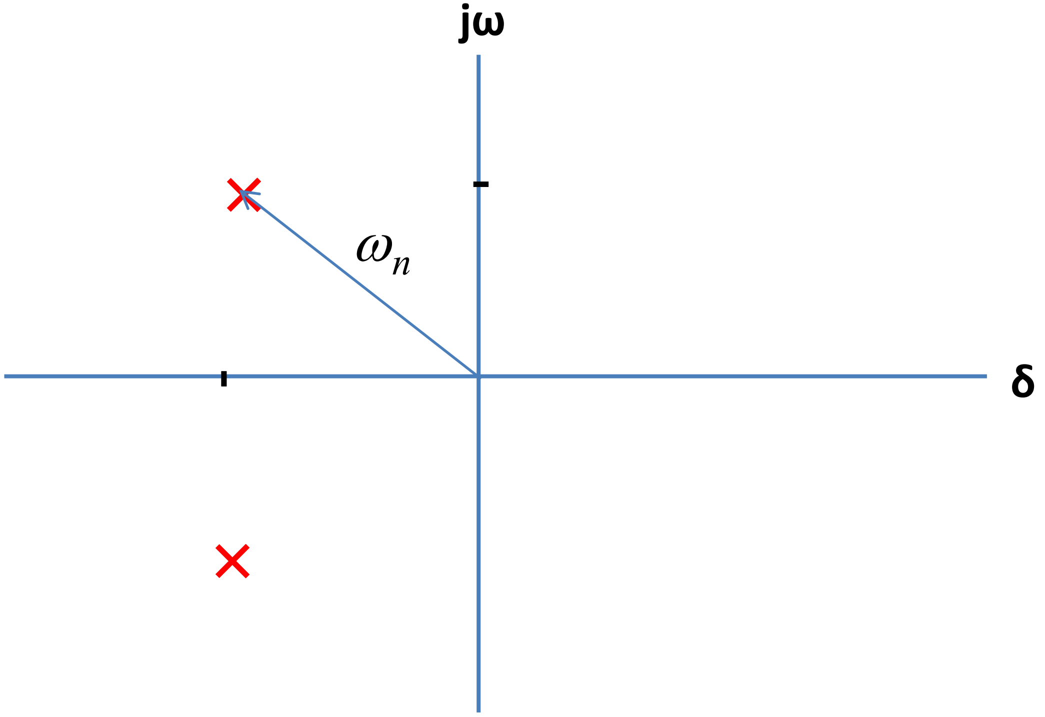
- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



S-Plane

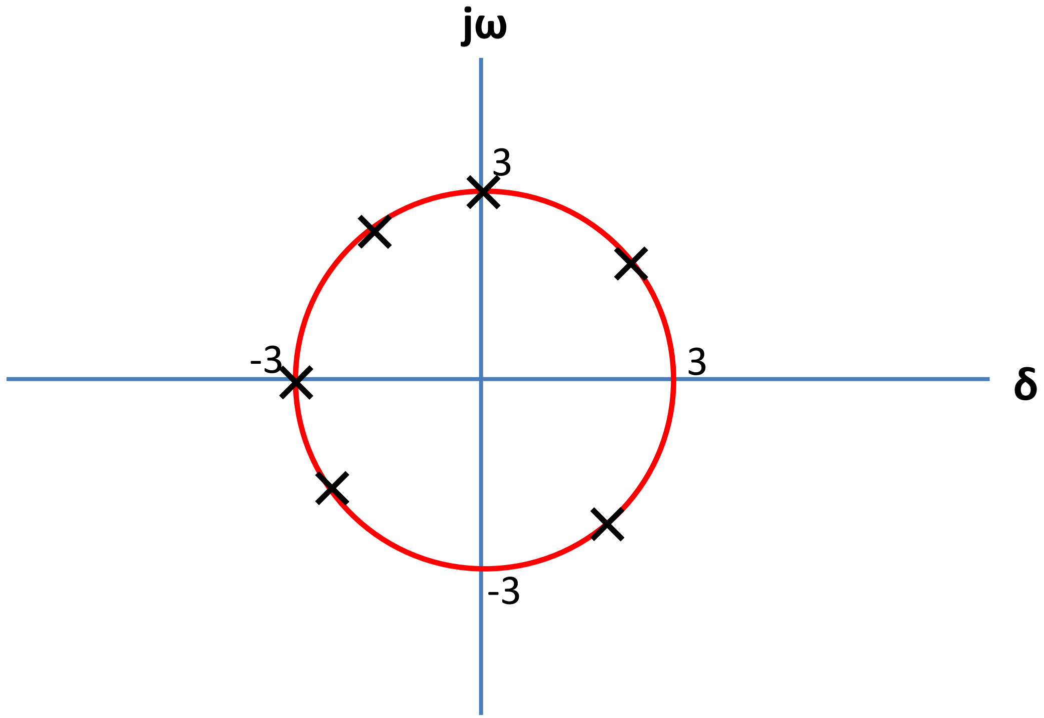
- Natural Undamped Frequency.

- Distance from the origin of s-plane to pole is natural undamped frequency in rad/sec.



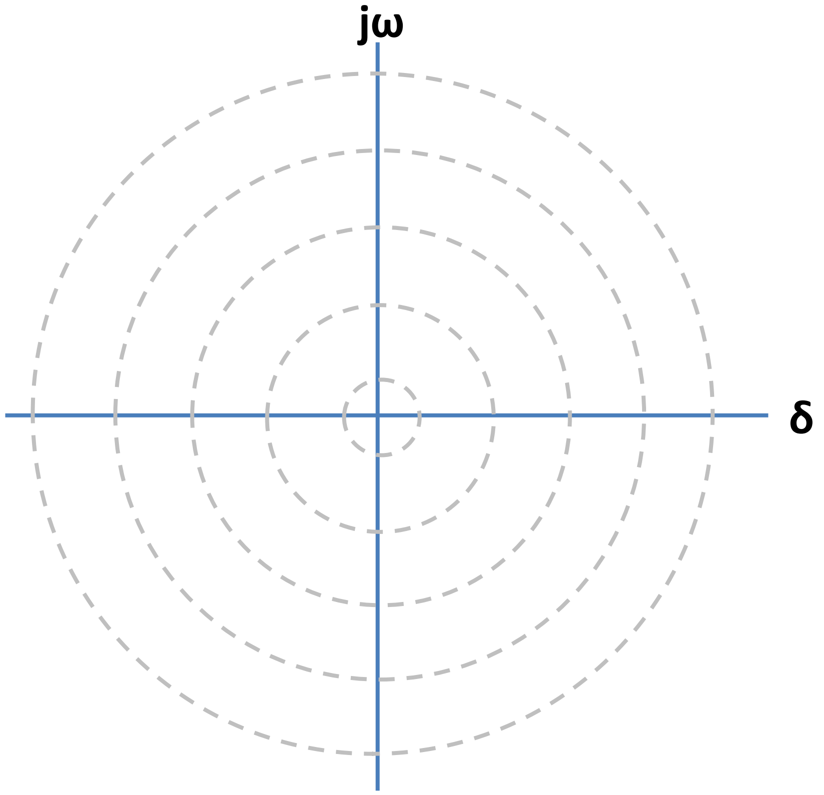
S-Plane

- Let us draw a circle of radius 3 in s-plane.
- If a pole is located anywhere on the circumference of the circle the natural undamped frequency would be *3 rad/sec*.



S-Plane

- Therefore the s-plane is divided into Constant Natural Undamped Frequency (ω_n) Circles.

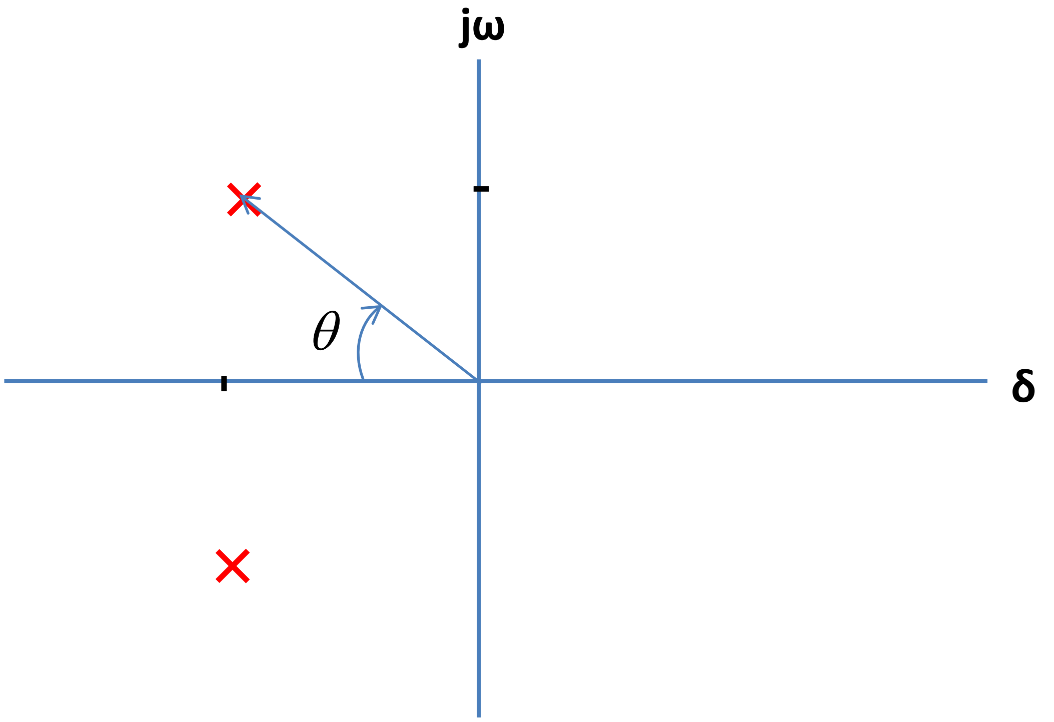


S-Plane

- Damping ratio.

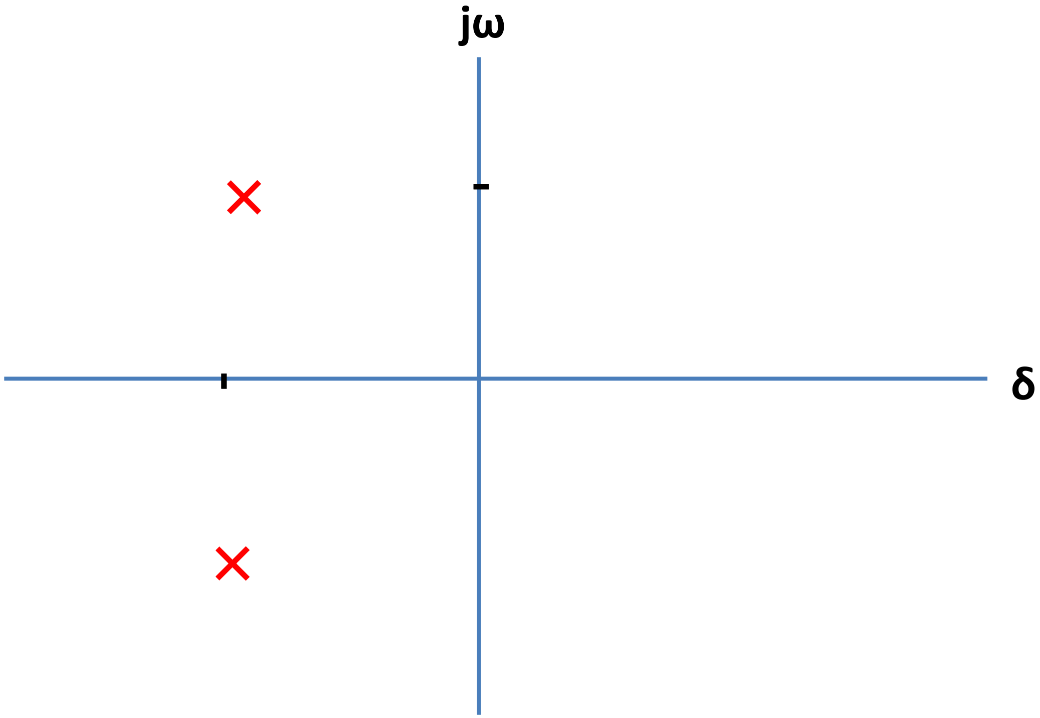
- Cosine of the angle between vector connecting origin and pole and –ve real axis yields damping ratio.

$$\zeta = \cos\theta$$



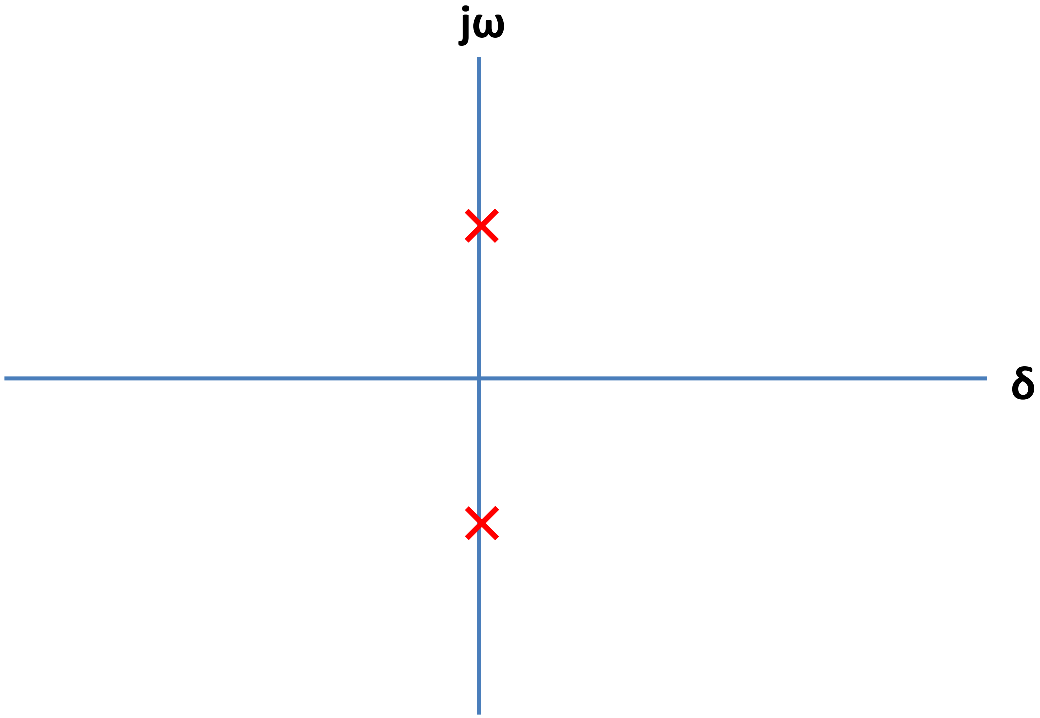
S-Plane

- For Underdamped system $0^\circ < \theta < 90^\circ$ therefore, $0 < \zeta < 1$



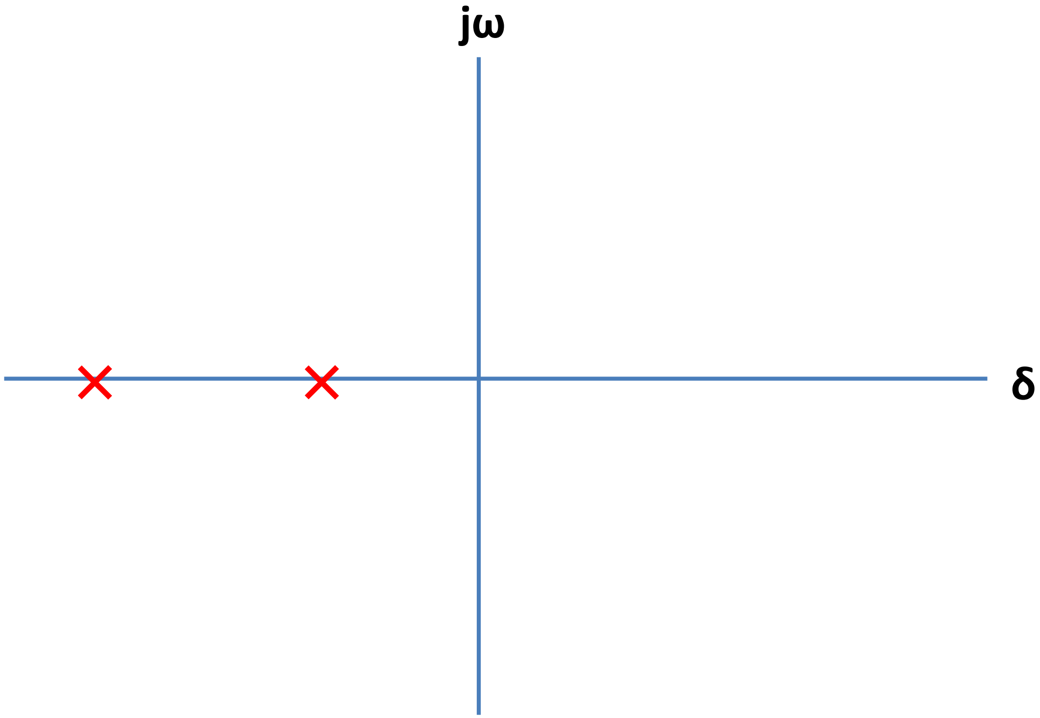
S-Plane

- For Undamped system $\theta = 90^\circ$ therefore, $\zeta = 0$



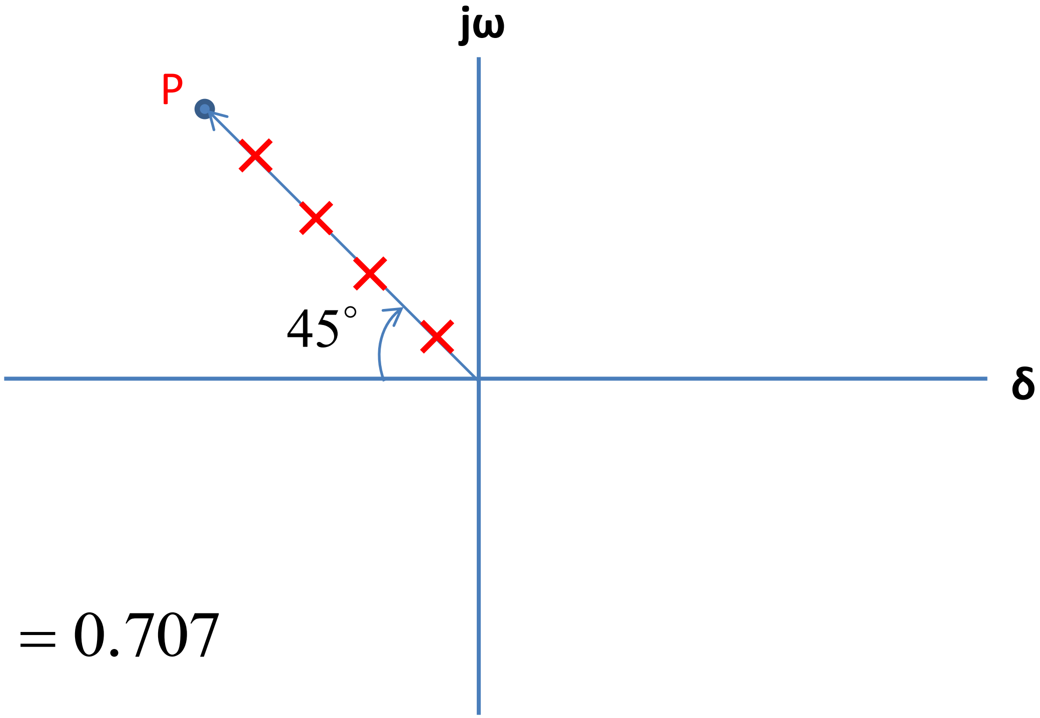
S-Plane

- For overdamped and critically damped systems $\theta = 0^\circ$ therefore, $\zeta \geq 0$



S-Plane

- Draw a vector connecting origin of s-plane and some point P.



$$\zeta = \cos 45^\circ = 0.707$$

S-Plane

- Therefore, s-plane is divided into sections of constant damping ratio lines.