Second – Order System

- *Second-order systems* exhibit a wide range of responses which must be analyzed and described.
  - Whereas for a *first-order system*, varying a single parameter changes the speed of response, changes in the parameters of a *second order system* can change the form of the response.

- *For example*: a second-order system can display characteristics much like a first-order system or, depending on component values, display damped or *pure oscillations* for its *transient response*.

Second – Order System

- A general second-order system is characterized by the following transfer function:

$$G(s) = \frac{b}{s^2 + as + b}$$

- We can re-write the above transfer function in the following form (closed loop transfer function):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second – Order System

ω_n $(\omega_n = \sqrt{b})$ - referred to as *the un-damped natural frequency* of the second order system, which is the frequency of oscillation of the system without damping.

ζ $(\zeta = \frac{a}{2\sqrt{b}})$ - referred to as *the damping ratio* of the second order system, which is a measure of the degree of resistance to change in the system output.

Poles;

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

Poles are complex if $\zeta < 1$!

Second – Order System

- According the value of ζ, a second-order system can be set into one of the four categories:

1. *Overdamped* - when the system has two real distinct poles ($\zeta > 1$).
2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)
3. *Undamped* - when the system has two imaginary poles ($\zeta = 0$).
4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).

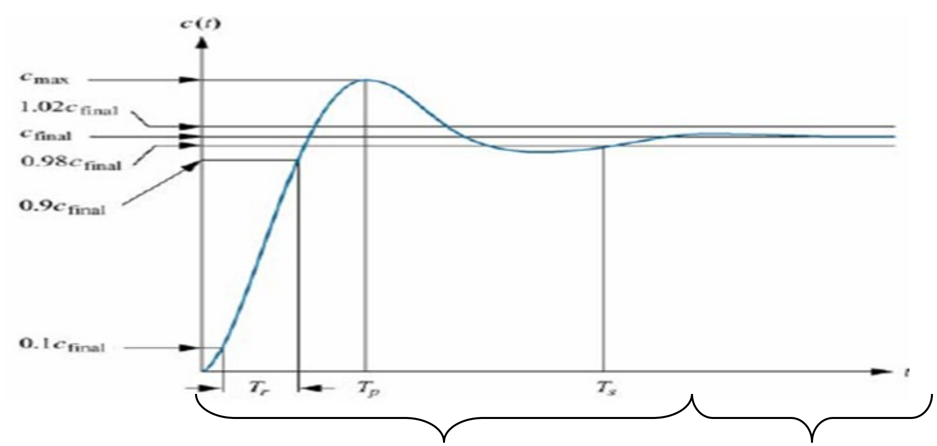
Time-Domain Specification

Given that the closed loop TF

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^{\;2}}{s^2 + 2\varsigma\omega_n s + \omega_n^{\;2}}$$

The system (2nd order system) is parameterized by ς and ω_n

For $0 < \varsigma < 1$ and $\omega_n > 0$, we like to investigate its response due to a unit step input



Transient Steady State

Two types of responses that are of interest:
(A)Transient response
(B)Steady state response

(A) For transient response, we have 4 specifications:

(a) T_r – rise time = $\dfrac{\pi - \theta}{\omega_n \sqrt{1-\varsigma^2}}$

(b) T_p – peak time = $\dfrac{\pi}{\omega_n \sqrt{1-\varsigma^2}}$

(c) %MP – percentage maximum overshoot = $e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} x100\%$

(d) T_s – settling time (2% error) = $\dfrac{4}{\varsigma\omega_n}$

(B) Steady State Response

(a) Steady State error

Question : How are the performance related to ς and ω_n ?

- Given a step input, i.e., $R(s) = 1/s$, then the system output (or step response) is;

$$C(s) = R(s)G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

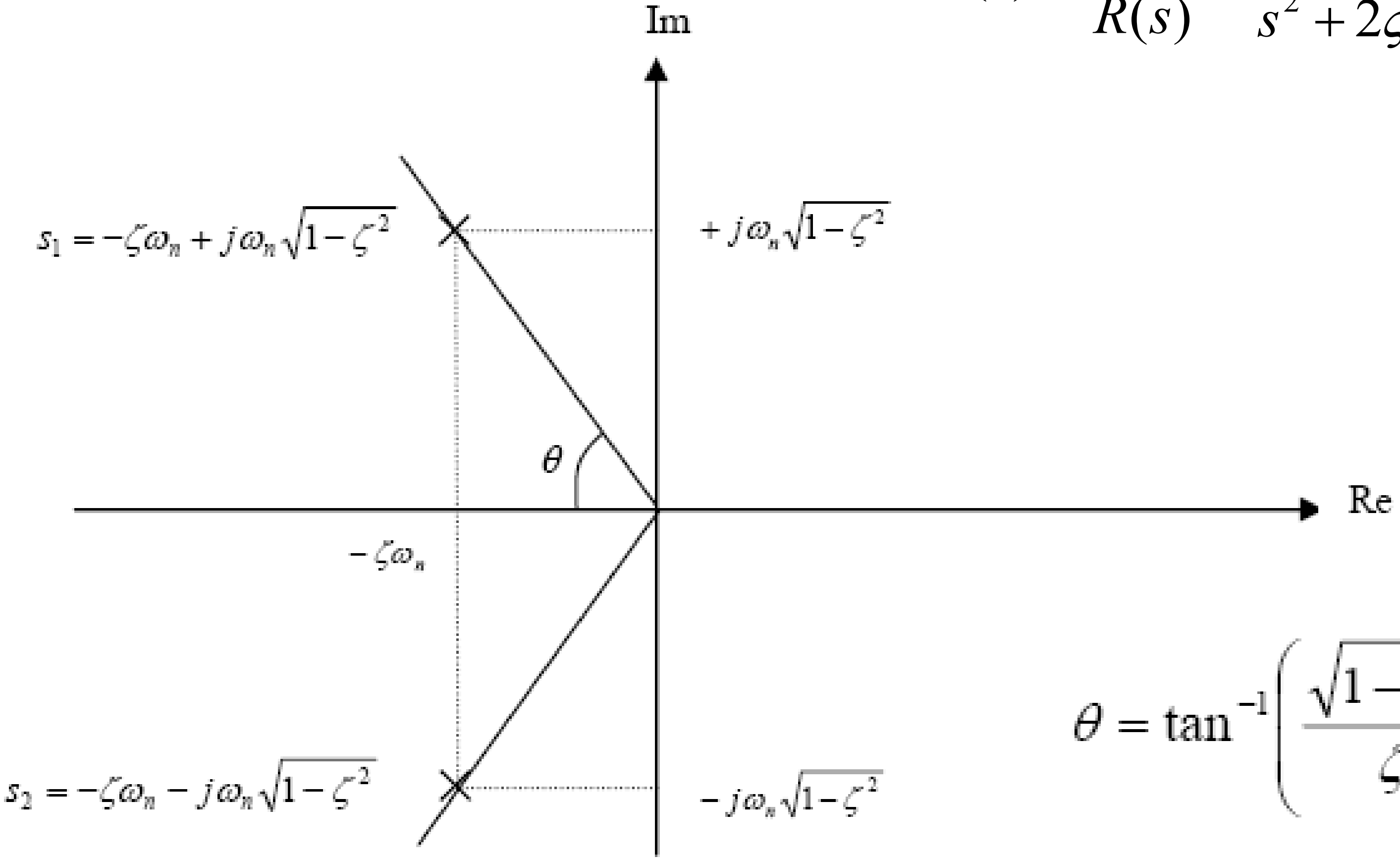
- Taking inverse Laplace transform, we have the step response;

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)$$

Where; $\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$; or $\theta = \cos^{-1}(\xi)$

Second – Order System

$$T(s)=\frac{C(s)}{R(s)}=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$$



$$\theta=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right).$$

Mapping the poles into s-plane

Lets re-write the equation for c(t):

Let: $\beta = \sqrt{1-\xi^2}$

and

$\omega_d = \omega_n \sqrt{1-\xi^2}$ } Damped natural frequency

$\omega_n > \omega_d$

Thus:

$$c(t) = 1 - \frac{1}{\beta} e^{-\xi \omega_n t} \sin\left(\omega_d t + \theta\right)$$

where $\theta = \cos^{-1}(\xi)$

Transient Response Analysis

1) Rise time, T_r. Time the response takes to rise from 0 to 100%

$$c(t)\Big|_{t=T_r} = 1 - \underbrace{\frac{1}{\beta} e^{-\xi\omega_n t}}_{\neq 0} \underbrace{\sin\left(\omega_d t + \theta\right)}_{=0} = 1$$

$$\sin(\omega_d T_r + \theta) = 0$$

$$\omega_d T_r + \theta = \sin^{-1}(0) = \pi$$

$$T_r = \frac{\pi - \theta}{\omega_n \sqrt{1 - \xi^2}}$$

Transient Response Analysis

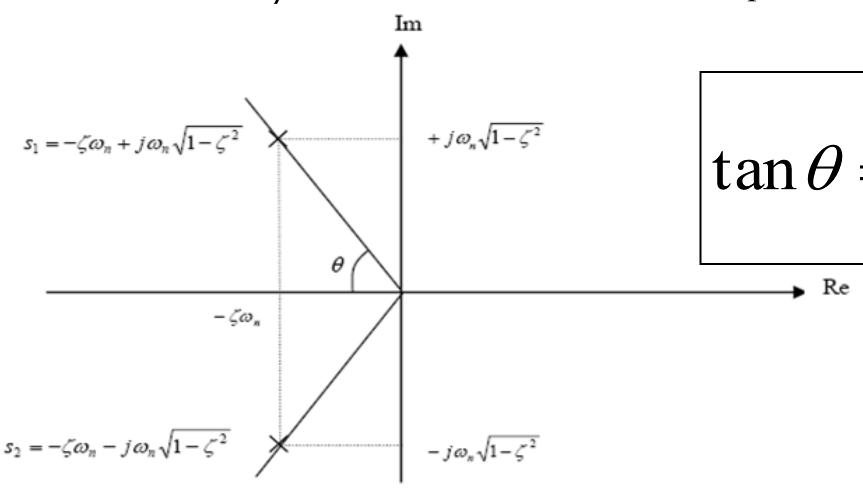
2) Peak time, T_p - The peak time is the time required for the response to reach the first peak, which is given by;

$$\left.\dot{c}(t)\right|_{t=T_p} = 0$$

$$\left.\dot{c}(t)\right|_{t=T_p} = -\frac{1}{\beta}(-\varsigma\omega_n)e^{-\varsigma\omega_n t}\sin(\omega_d t+\theta) - \tfrac{1}{\beta}e^{-\varsigma\omega_n t}\cos(\omega_d t+\theta)\left[\omega_n\sqrt{1-\varsigma^2}\right] = 0$$

$$\frac{\varsigma\omega_n}{\beta}e^{-\varsigma\omega_n T_p}\sin(\omega_d T_p+\theta) = \frac{\left[\omega_n\sqrt{1-\varsigma^2}\right]}{\beta}e^{-\varsigma\omega_n T_p}\cos(\omega_d T_p+\theta)$$

$$\tan(\omega_d T_p+\theta) = \frac{\sqrt{1-\varsigma^2}}{\varsigma}$$



$$\tan\theta = \frac{\sqrt{1-\varsigma}}{\varsigma}$$

We know that $\tan(\theta) = \tan(\pi + \theta)$

So, $\tan(\omega_d T_p + \theta) = \tan(\pi + \theta)$

From this expression:

$$\omega_d T_p + \theta = \pi + \theta$$

$$\omega_d T_p = \pi$$

$$\boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \varsigma^2}}}$$

Transient Response Analysis

3) Percent overshoot, %OS - The percent overshoot is defined as the amount that the waveform at the peak time overshoots the steady-state value, which is expressed as a percentage of the steady-state value.

$$\%MP \equiv \frac{C(T_p) - C(\infty)}{C(\infty)} x100\%$$

OR

$$\%OS = \frac{C\max - Cfinal}{Cfinal}\ x\ 100$$

$$\frac{C(T_p)-1}{1}x100\% = -\frac{1}{\beta}e^{-\xi\omega_n t}\sin\left(\omega_d t+\theta\right)x100\%$$

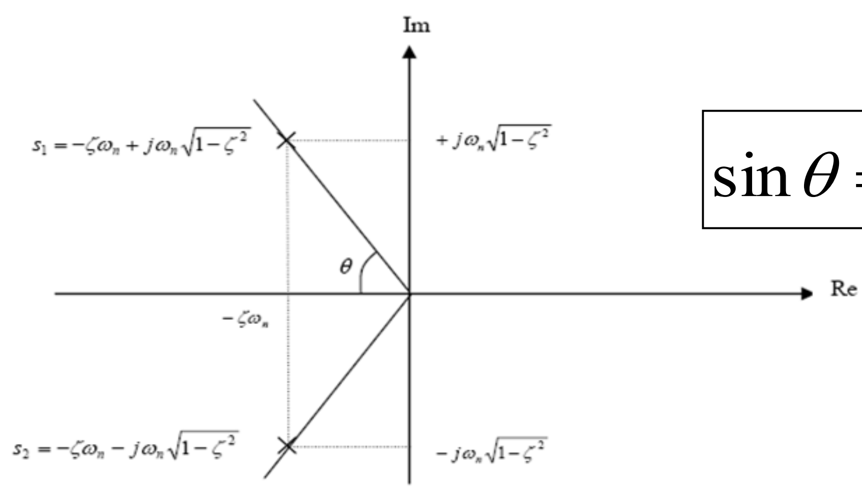
$$= -\frac{1}{\beta}e^{-\xi\omega_n\left[\frac{\pi}{\omega_n\sqrt{1-\varsigma^2}}\right]}\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right)+\theta\right)x100\%$$

$$= -\frac{1}{\beta}e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}\sin\left(\pi+\theta\right)x100\%$$

$$= \frac{\sin(\theta)}{\beta}e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}x100\% = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}x100\%$$

From slide 24

$$\beta = \sqrt{1-\xi^2}$$



$$\boxed{\sin\theta = \sqrt{1-\varsigma^2}}$$

Therefore,

$$\%MP = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} x100\%$$

- For given %*OS*, the damping ratio can be solved from the above equation;

$$\varsigma = \frac{-\ln(\%MP/100)}{\sqrt{\pi^2 + \ln^2(\%MP/100)}}$$

Transient Response Analysis

4) Setting time, T_s - The settling time is the time required for the amplitude of the sinusoid to decay to 2% of the steady-state value.

To find T_s, we must find the time for which c(t) reaches & stays within $\pm$2% of the steady state value, $c_{final.}$ The settling time is the time it takes for the amplitude of the decaying sinusoid in c(t) to reach 0.02, or

$$e^{-\varsigma\omega_n T_s}\frac{1}{\sqrt{1-\varsigma^2}}=0.02$$

Thus,

$$\boxed{T_s=\frac{4}{\varsigma\omega_n}}$$

UNDERDAMPED

Example 2: *Find the natural frequency and damping ratio for the system with transfer function*

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Solution:

Compare with general TF

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- $\omega_n = 6$
- $\xi = 0.35$

UNDERDAMPED

Example 3: Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

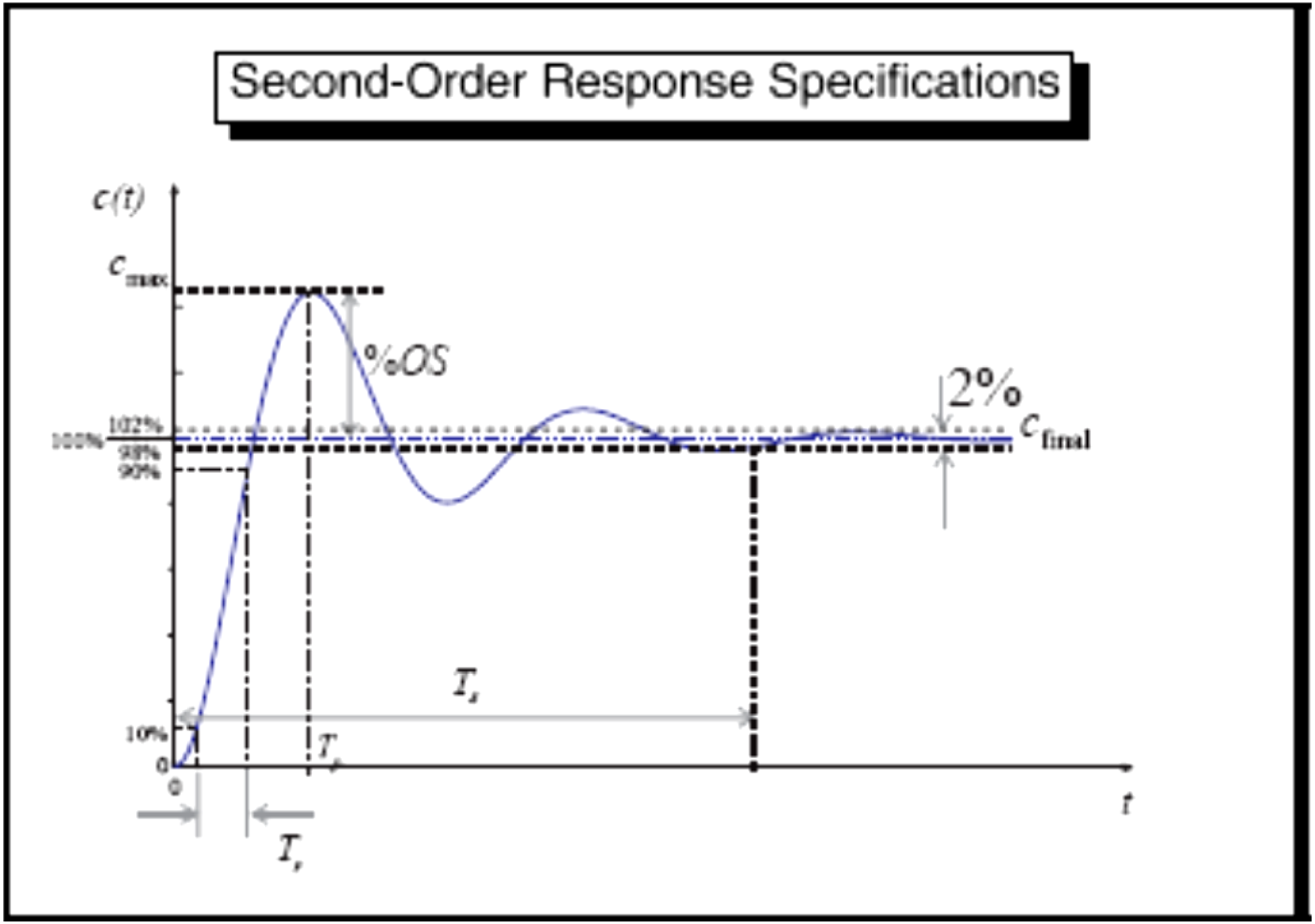
$$find \quad T_s,\ \%OS,\ T_p$$

Solution:

$$\omega_n = 10 \qquad \xi = 0.75$$

$$T_s = 0.533s,\ \%OS = 2.838\%,\ T_p = 0.475s$$

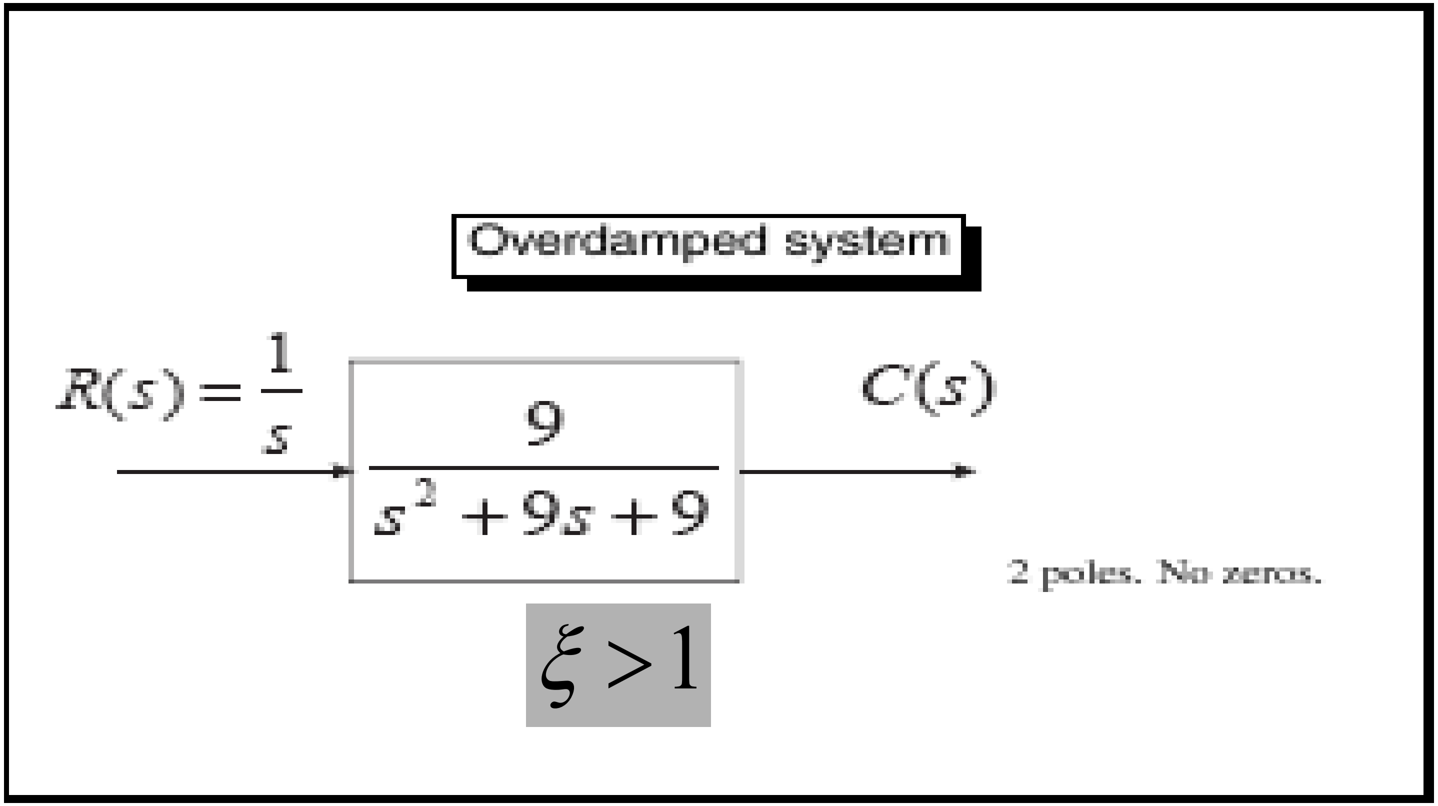
UNDERDAMPED



$$G(s) = \frac{b}{s^2 + as + b}$$

Overdamped Response

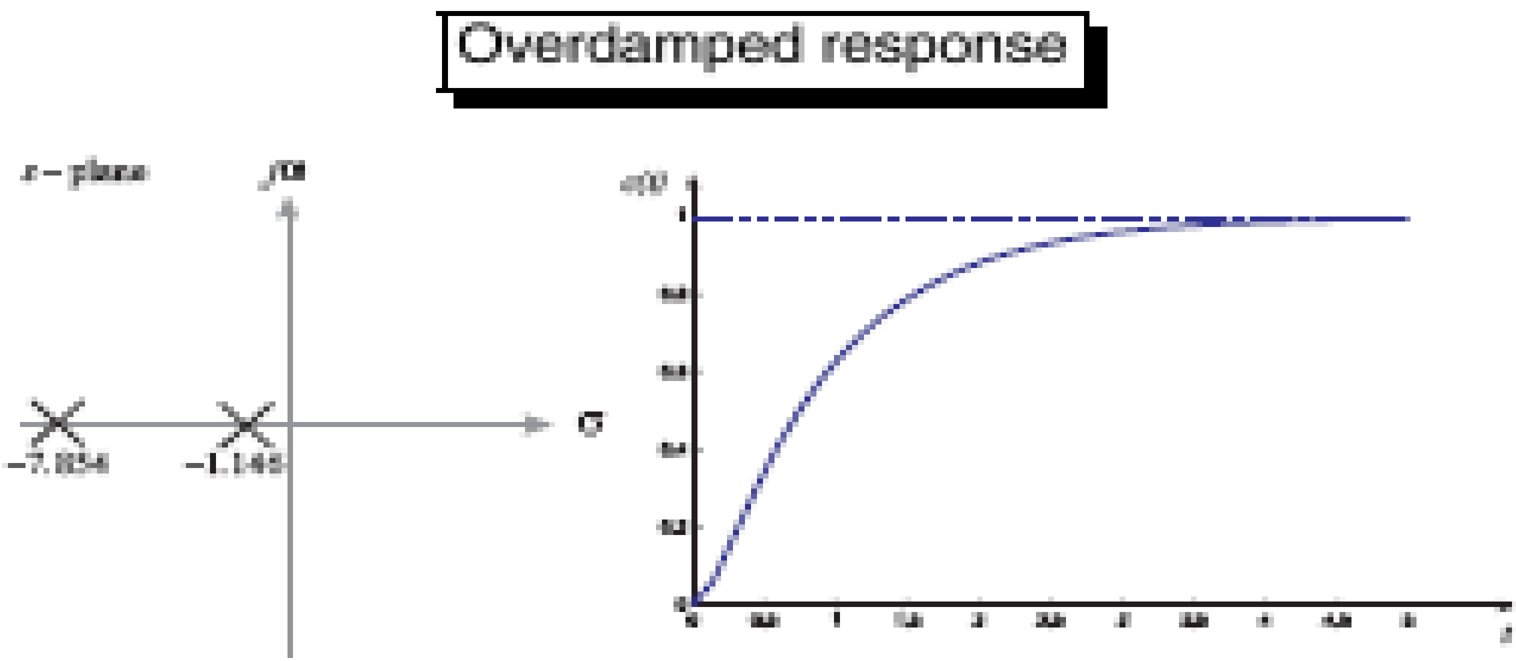
a = 9



$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

s= 0; s = -7.854; s = -1.146 *(two real poles)*

$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$

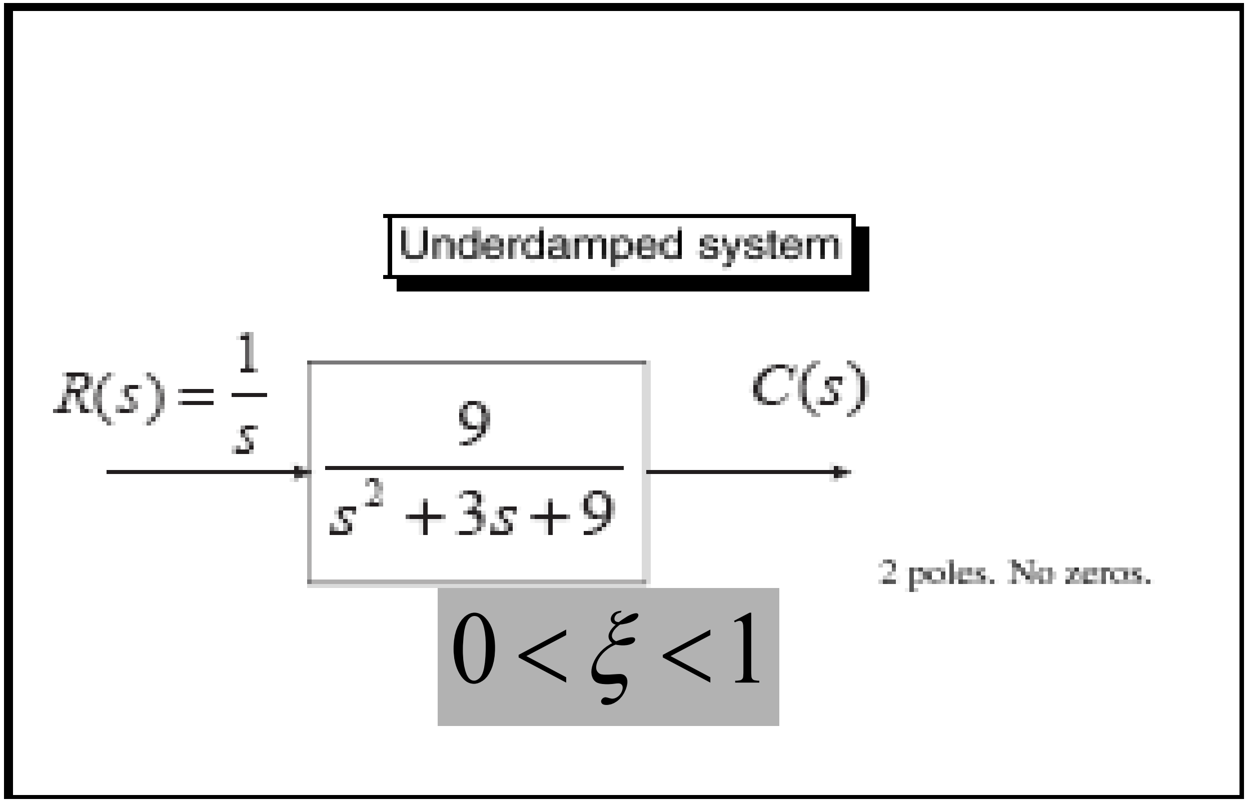


OVERDAMPED RESPONSE !!!

$$G(s) = \frac{b}{s^2 + as + b}$$

Underdamped Response

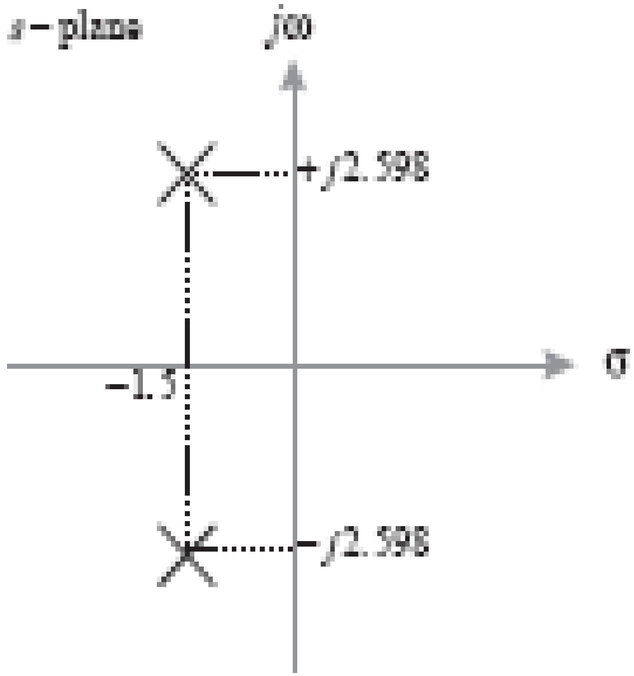
a = 3

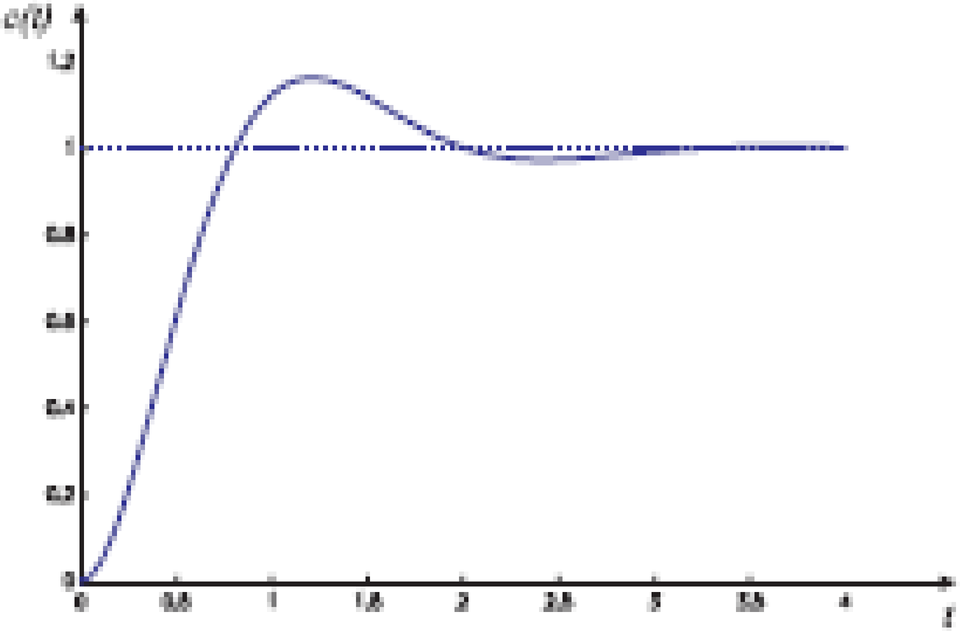


$$c(t) = K_1 + e^{-1.5t}(K_2 \cos 2.598t + K_3 \sin 2.598t)$$

s = 0; s = -1.5 ± j2.598 *(two complex poles)*

Underdamped response



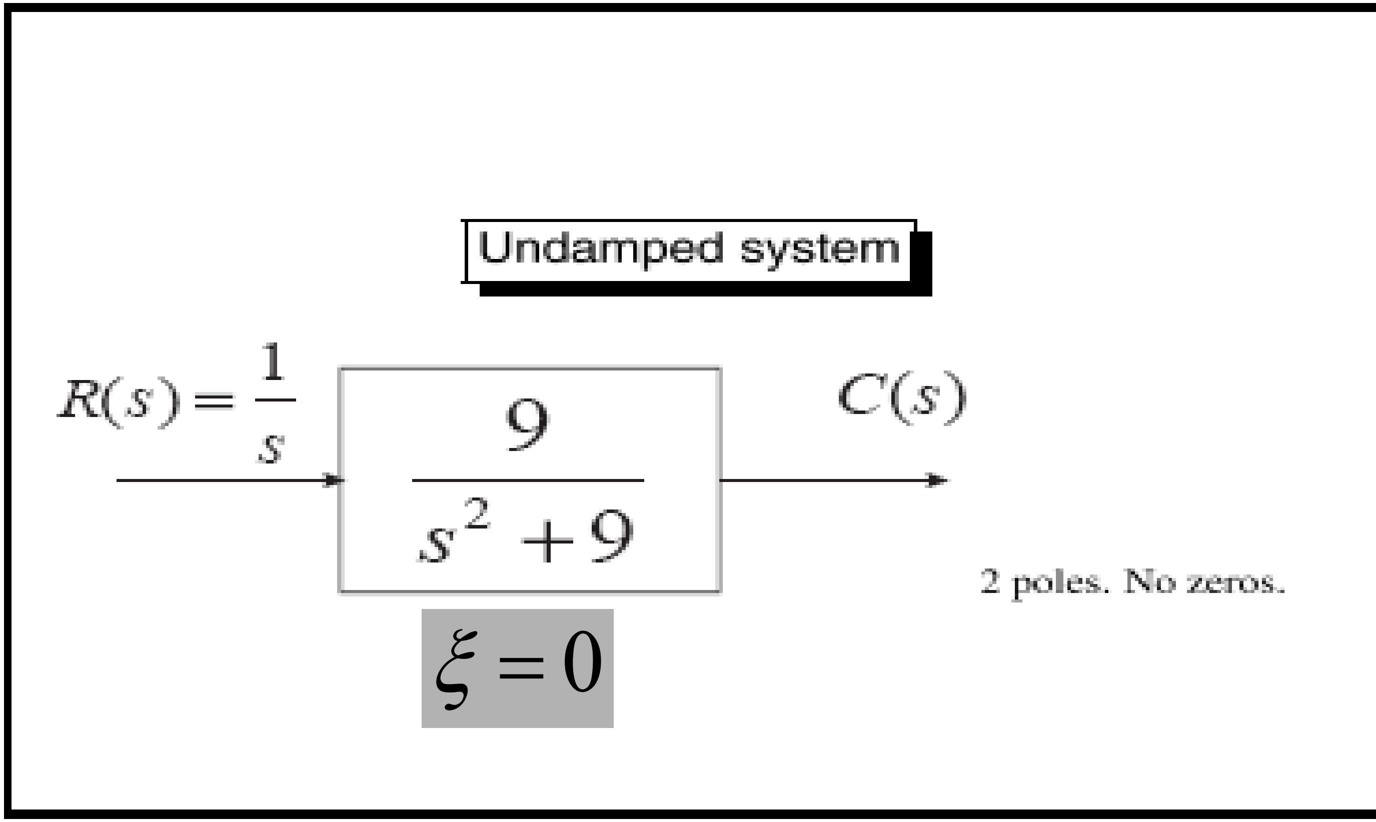


UNDERDAMPED RESPONSE !!!

Undamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

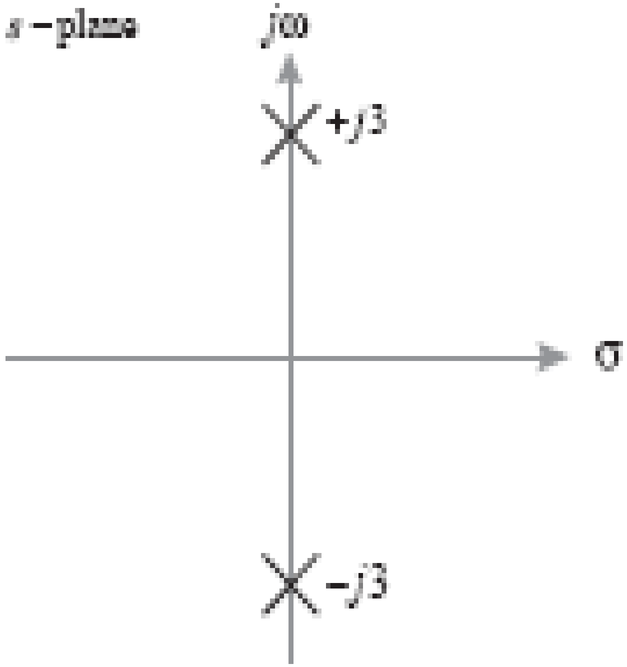
a = 0

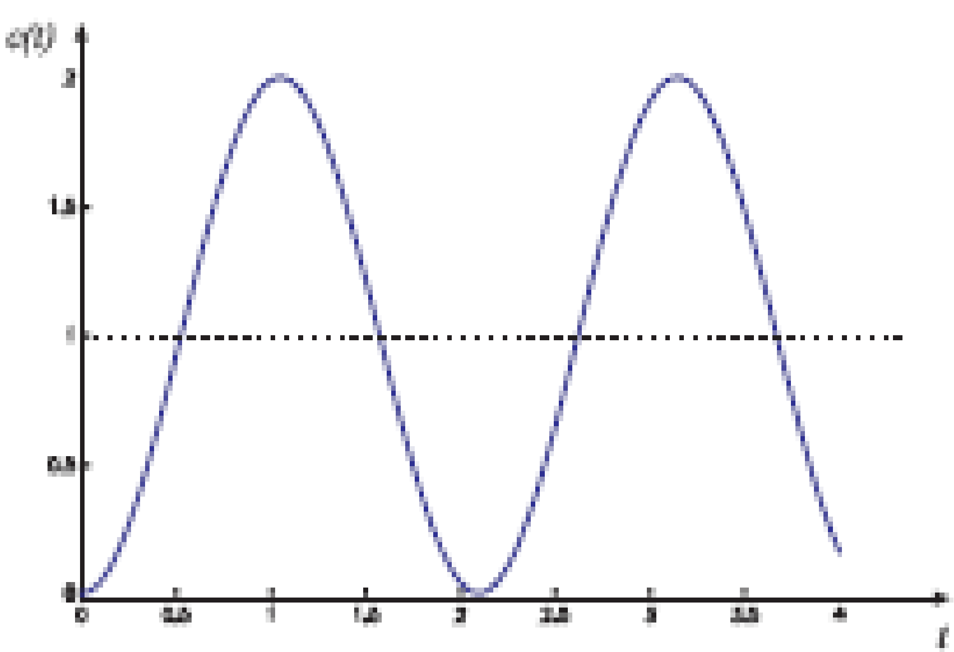


$$c(t) = K_1 + K_2 \cos 3t$$

s = 0; s = ± j3 *(two imaginary poles)*

Undamped response



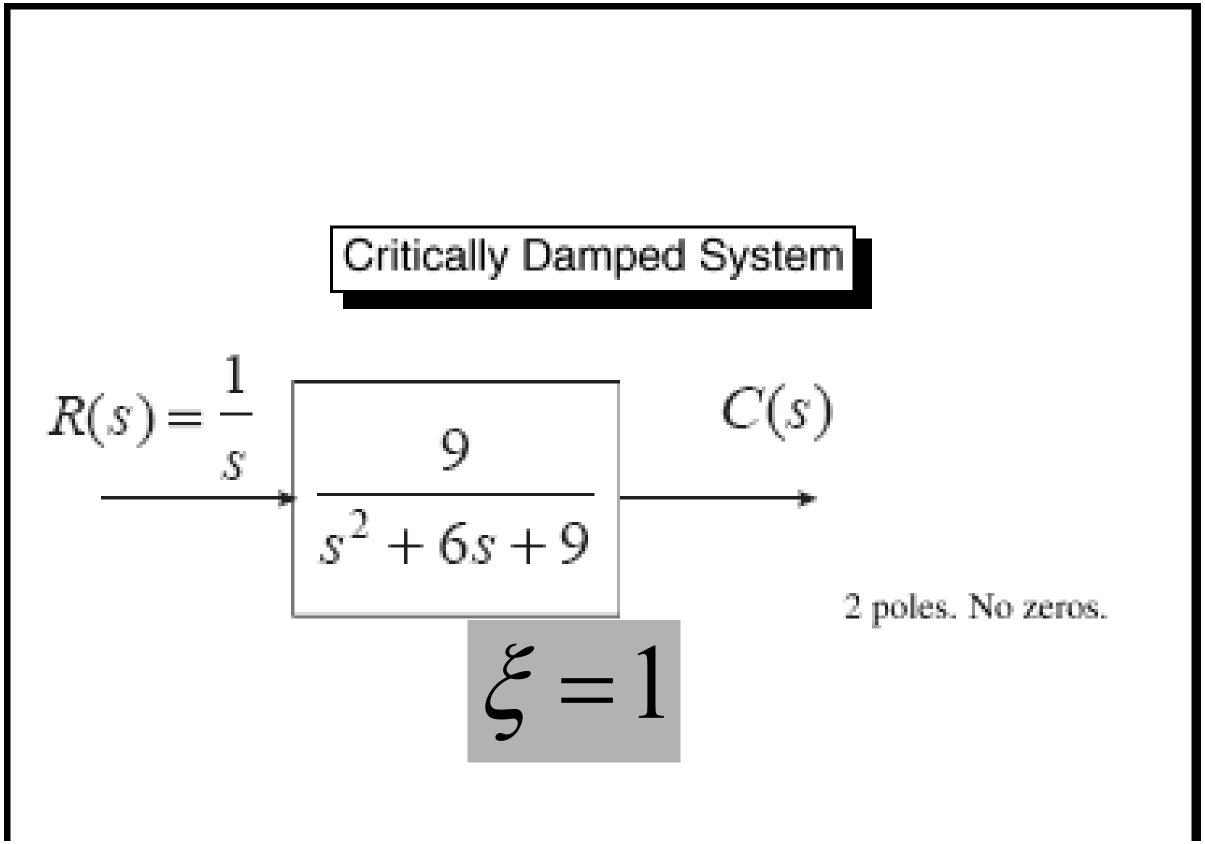


UNDAMPED RESPONSE !!!

Critically Damped System

$$G(s) = \frac{b}{s^2 + as + b}$$

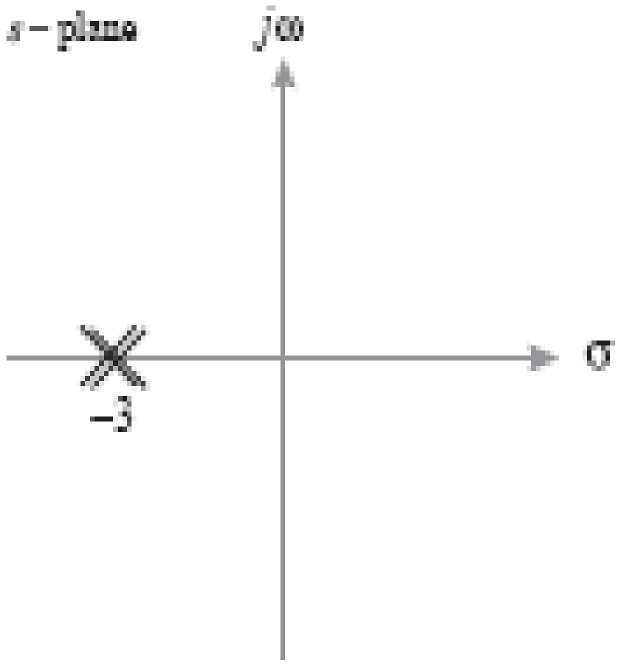
a = 6

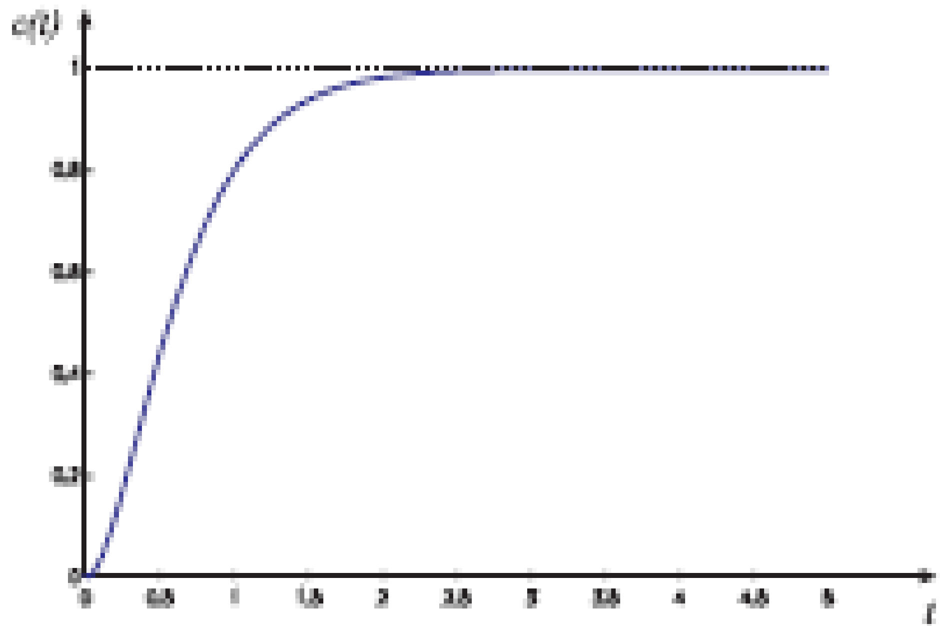


$$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$$

S = 0; s = -3,-3 *(two real and equal poles)*

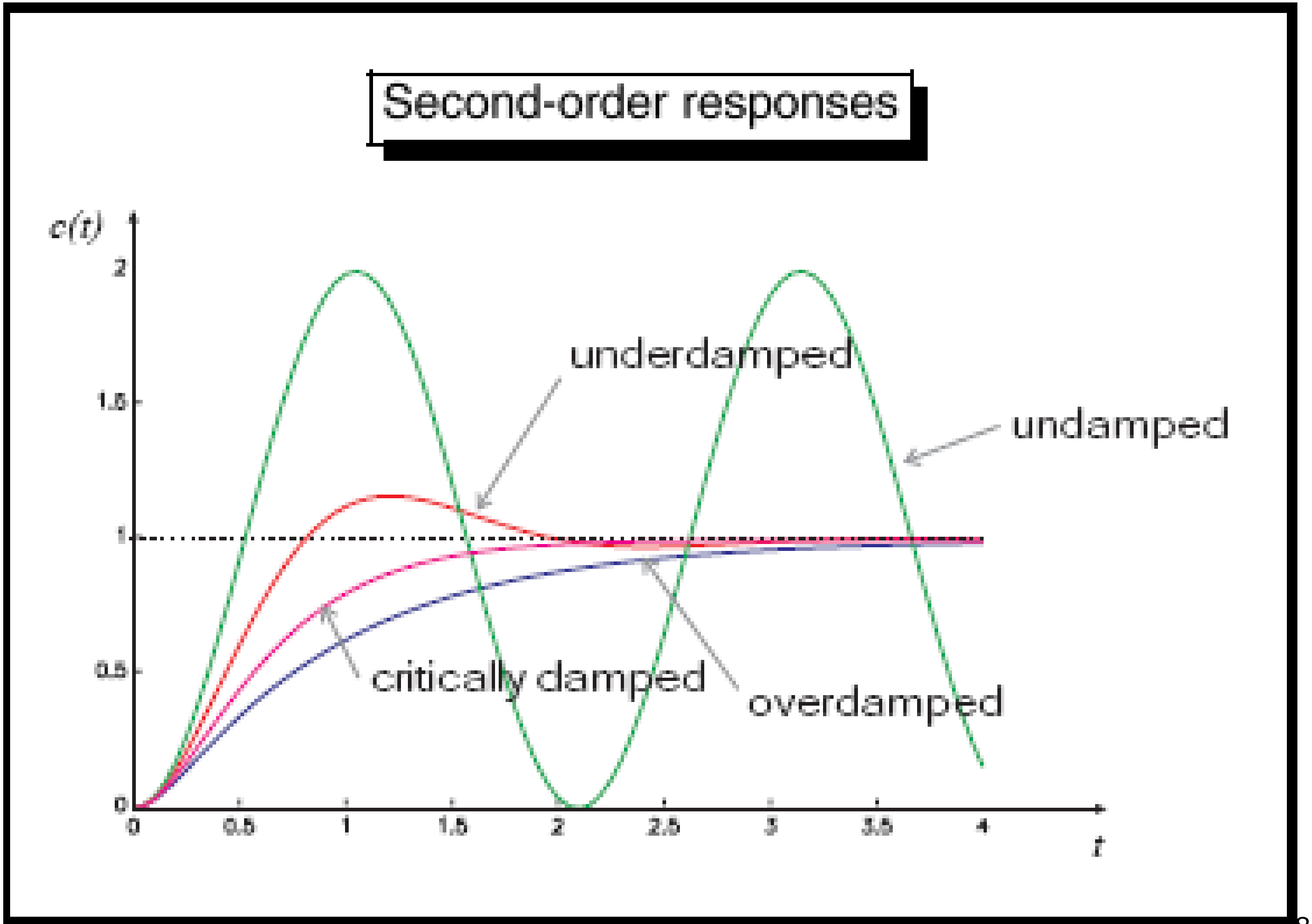
Critically Damped Response

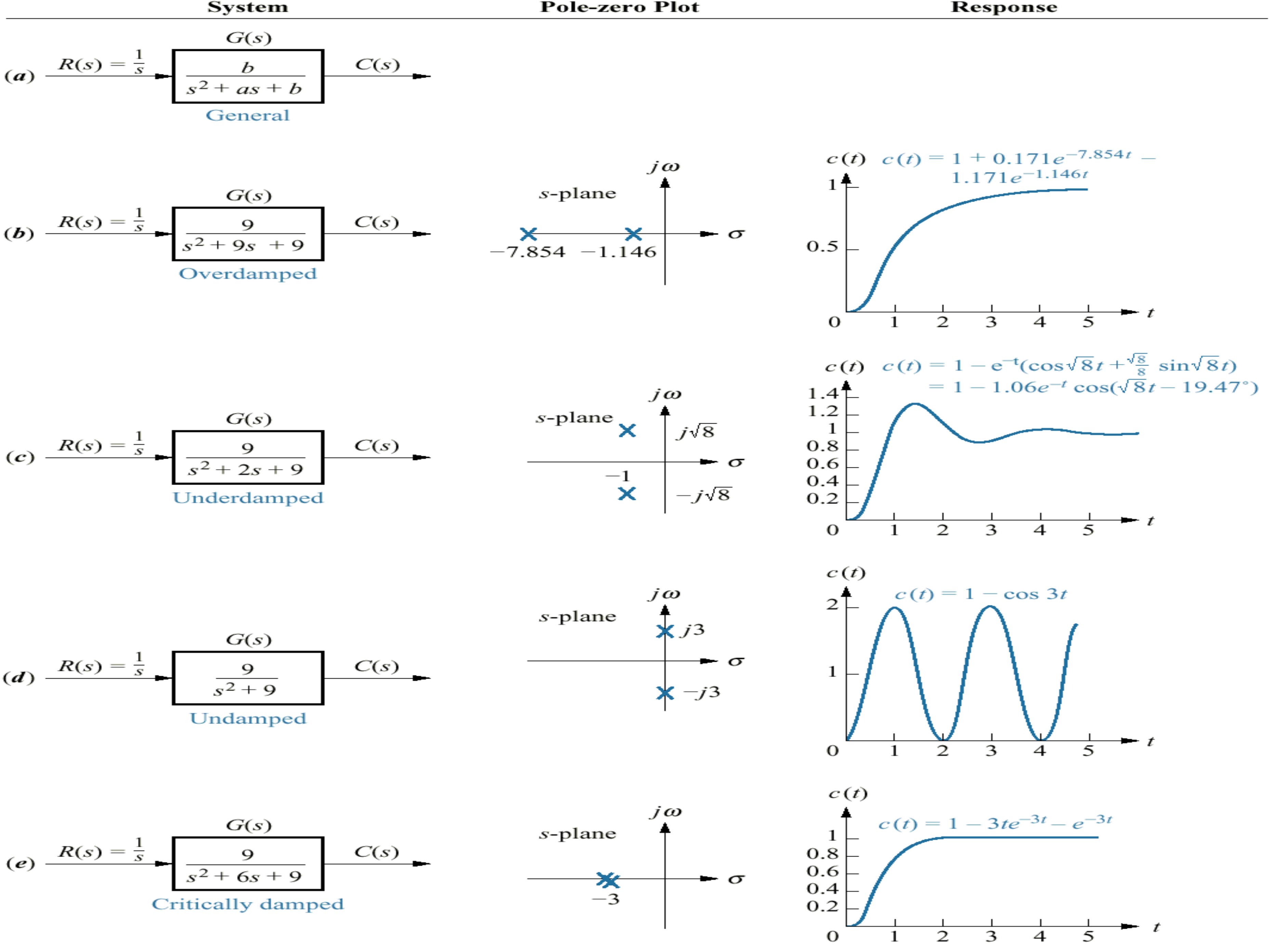




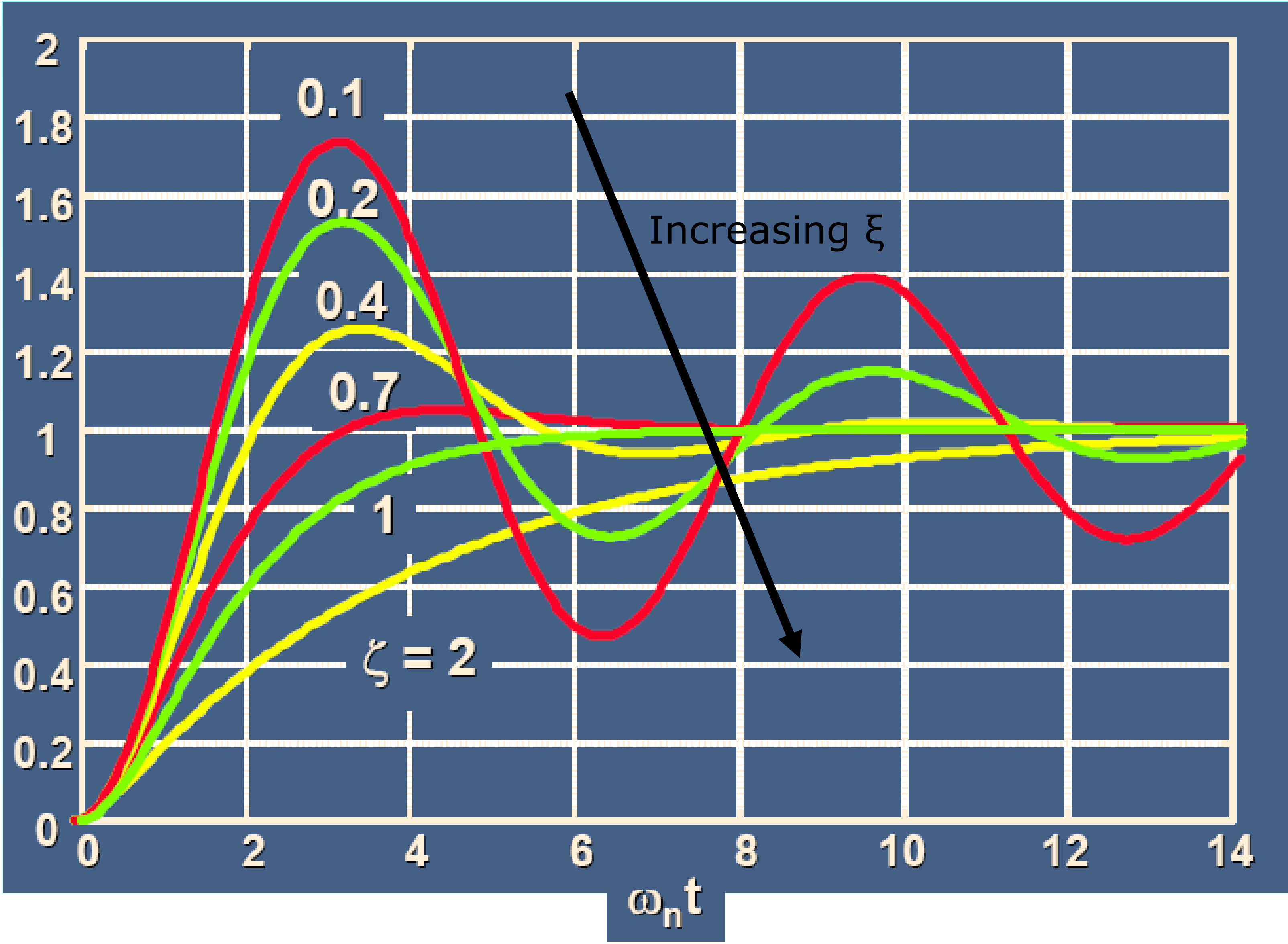
CRITICALLY DAMPED RESPONSE !!!

Second – Order System




System
Pole-zero Plot
Response
(a)
R(s) = 1/s
G(s)
b / (s^2 + as + b)
C(s)
General
(b)
R(s) = 1/s
G(s)
9 / (s^2 + 9s + 9)
C(s)
Overdamped
jω
s-plane
σ
−7.854
−1.146
c(t)
c(t) = 1 + 0.171e^{−7.854t} − 1.171e^{−1.146t}
1
0.5
0
1 2 3 4 5
t
(c)
R(s) = 1/s
G(s)
9 / (s^2 + 2s + 9)
C(s)
Underdamped
jω
s-plane
j√8
σ
−1
−j√8
c(t)
c(t) = 1 − e^{−t}(cos√8 t + (√8/8) sin√8 t)
= 1 − 1.06e^{−t} cos(√8 t − 19.47°)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5
t
(d)
R(s) = 1/s
G(s)
9 / (s^2 + 9)
C(s)
Undamped
jω
s-plane
j3
σ
−j3
c(t)
c(t) = 1 − cos 3t
2
1
0
1 2 3 4 5
t
(e)
R(s) = 1/s
G(s)
9 / (s^2 + 6s + 9)
C(s)
Critically damped
jω
s-plane
σ
−3
c(t)
c(t) = 1 − 3te^{−3t} − e^{−3t}
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5
t

Effect of different damping ratio, ξ



Second – Order System

Example 4: Describe the nature of the second-order system response via the value of the damping ratio for the systems with transfer function

1. $$G(s) = \frac{12}{s^2 + 8s + 12}$$

2. $$G(s) = \frac{16}{s^2 + 8s + 16}$$

3. $$G(s) = \frac{20}{s^2 + 8s + 20}$$

Do them as your own revision