Methods of Determining Stability

Routh-Hurwitz criteria and limitations.

Determining Stability

- For all practical purposes, there is no need to compute the complete system response to determine stability.
- When the system parameters are all know the roots of the characteristic equation can be found using various methods including the Automatic Control Systems Software (ACSYS) in conjunction with MATLAB Toolbox.

Determining Stability

- What are some of the methods for determining stability in control engineering?

Methods for Determining Stability

- **Routh–Hurwitz Criterion** – this is an algebraic method that provides information on the absolute stability of a system that has as characteristic equation with constant coefficients. The criterion tests whether any of the roots of the characteristic equation lie in the right half s-plane.
- **Nyquist Criterion** – this is a semi graphical method that gives information on the difference between the number of poles and zeros of the closed loop transfer function that are in the right – half s-plane by observing the behavior of the Nyquist plot of the loop transfer function.

Methods for Determining Stability

- **Bode Diagram** – this is the diagram plot of the magnitude of the loop transfer function G(jω) H (jω) in dB and the phase of G (jω) H (jω) in degrees, all verses frequency ω.

- The stability of the closed loop system can be determined by observing the behavior of these plots.

Tip Off

- It will be evident throughout this course, that most of the design and analysis techniques on control systems will represent alternative methods of solving the same problem.

- The designer simply has to choose the best analytical tool, depending on the particular situation.

Tip Off

- In the course of normal system design, it is not merely necessary that the system be stable.
- It is essential that the system be sufficiently stable that transient disturbances will decay quickly enough to permit rapid recovery by the controlled variable.
- In the next sections, however, the major emphasis is on concepts and methods of determining system stability.

Routh-Hurwitz Criterion

- What is RHC Analysis?
- The Routh–Hurwitz Criterion (RHC) represents a method of determining the location of zeros of a polynomial with constant real coefficients with respect to the left half and right half of the s-plane, without actually solving for the zeros.

Routh-Hurwitz Stability Criterion

The Routh-Hurwitz criterion is a method for determining whether a linear system is stable or not by examining the locations of the roots of the characteristic equation of the system. In fact, the method determines only if there are roots that lie outside of the left half plane; it does not actually compute the roots. Consider the characteristic equation

$$1+GH(s)=D(s)=a_n s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0=0$$

To determine whether this system is stable or not, check the following conditions:

1. Two necessary but not sufficient conditions that all the roots have negative real parts are

   a) All the polynomial coefficients must have the same sign.
   b) All the polynomial coefficients must be nonzero.

2. If condition (1) is satisfied, then compute the Routh-Hurwitz array as follows:

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & & \cdots \\
s^{n-4} & & & \vdots & &
\end{array}$$

2. If condition (1) is satisfied, then compute the Routh-Hurwitz array as follows:

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & & \cdots \\
s^{n-4} & & & \vdots & & \\
\vdots & & & \vdots & & \\
s^1 & & & \vdots & & \\
s^0 & & & \vdots & &
\end{array}
$$

where the a_i's are the polynomial coefficients, and the coefficients in the rest of the table are computed using the following pattern:

$$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{-1}{a_{n-1}}(a_n a_{n-3} - a_{n-2} a_{n-1}) \qquad b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

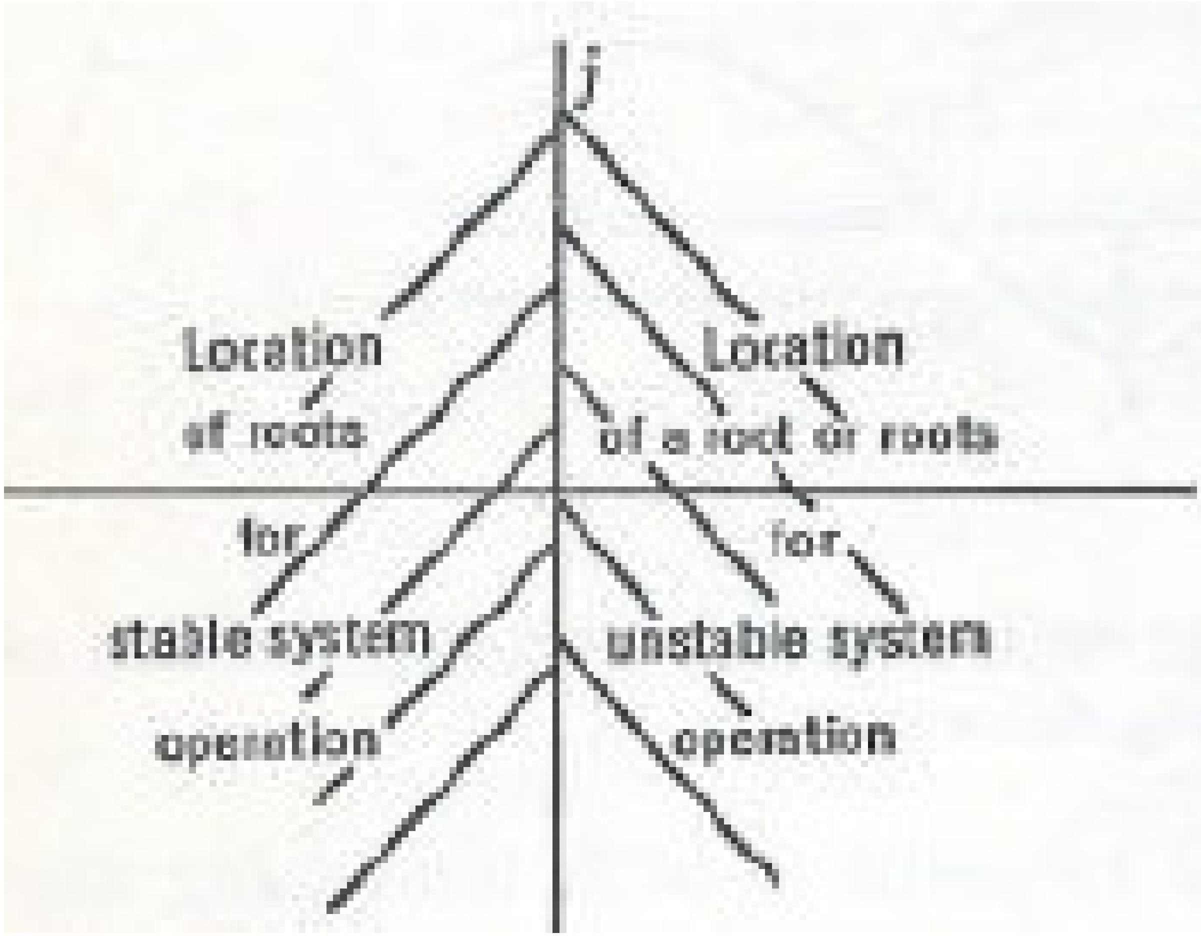
$$b_3 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix} \quad \cdots \qquad c_1 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \qquad c_2 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} \qquad \cdots$$

3. The necessary condition that all roots have negative real parts is that all the elements of the first column of the array have the same sign. The number of changes of sign equals the number of roots with positive real parts.

4. Special Case 1: The first element of a row is zero, but some other elements in that row are nonzero. In this case, simply replace the zero element by "ε", complete the table development, and then interpret the results assuming that "ε" is a small number of the same sign as the element above it. The results must be interpreted in the limit as $\varepsilon \to 0$.

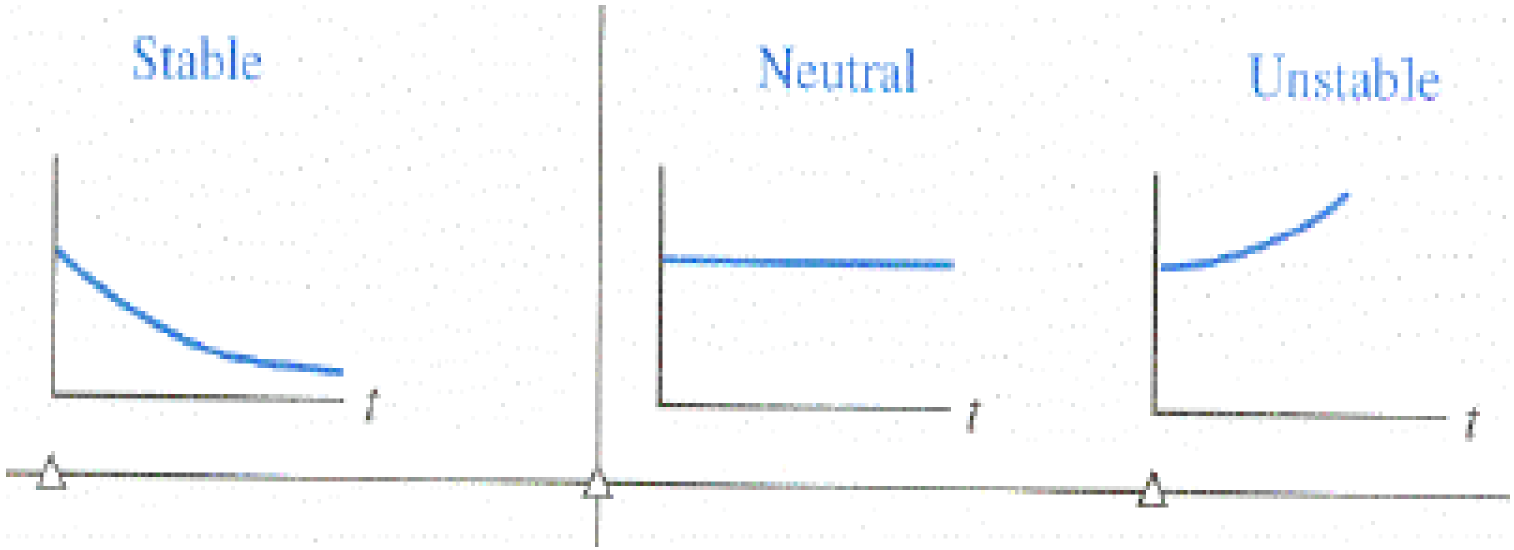
5. Special Case 2: All the elements of a particular row are zero. In this case, some of the roots of the polynomial are located symmetrically about the origin of the *s*-plane, e.g., a pair of purely imaginary roots. The zero row will always occur in a row associated with an odd power of *s*. The row just above the zero row holds the coefficients of the auxiliary polynomial. The roots of the auxiliary polynomial are the symmetrically placed roots. Be careful to remember that the coefficients in the array skip powers of *s* from one coefficient to the next.

Reminder



jω
Location
of roots
for
stable system
operation
Location
of a root or roots
for
unstable system
operation

Output Signals Characteristic for Stability, Neutral & Unstable



Stability Condition

- **A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.**

Routh-Hurwitz Criterion

- The Routh–Hurwitz Criterion (RHC) represents a method of determining the location of zeros of a polynomial with constant real coefficients with respect to the left half and right half of the s-plane, without actually solving for the zeros.

Routh-Hurwitz Criterion Background

- The stability analysis has occupied the interest of many engineers including Maxwell and Vishnegradsky.
- In the late 1800s, A. Hurwitz and E.J. Routh independently published a method of investigating the stability of a linear system.
- The Routh-Hurwitz stability method provides an answer to the question of stability by considering the characteristic equation of the system.

Routh-Hurwitz Criterion

- The Routh-Hurwitz criterion is based on ordering the coefficients of the characteristic equation:

$$\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

Routhian Array

- Into an array as follows:

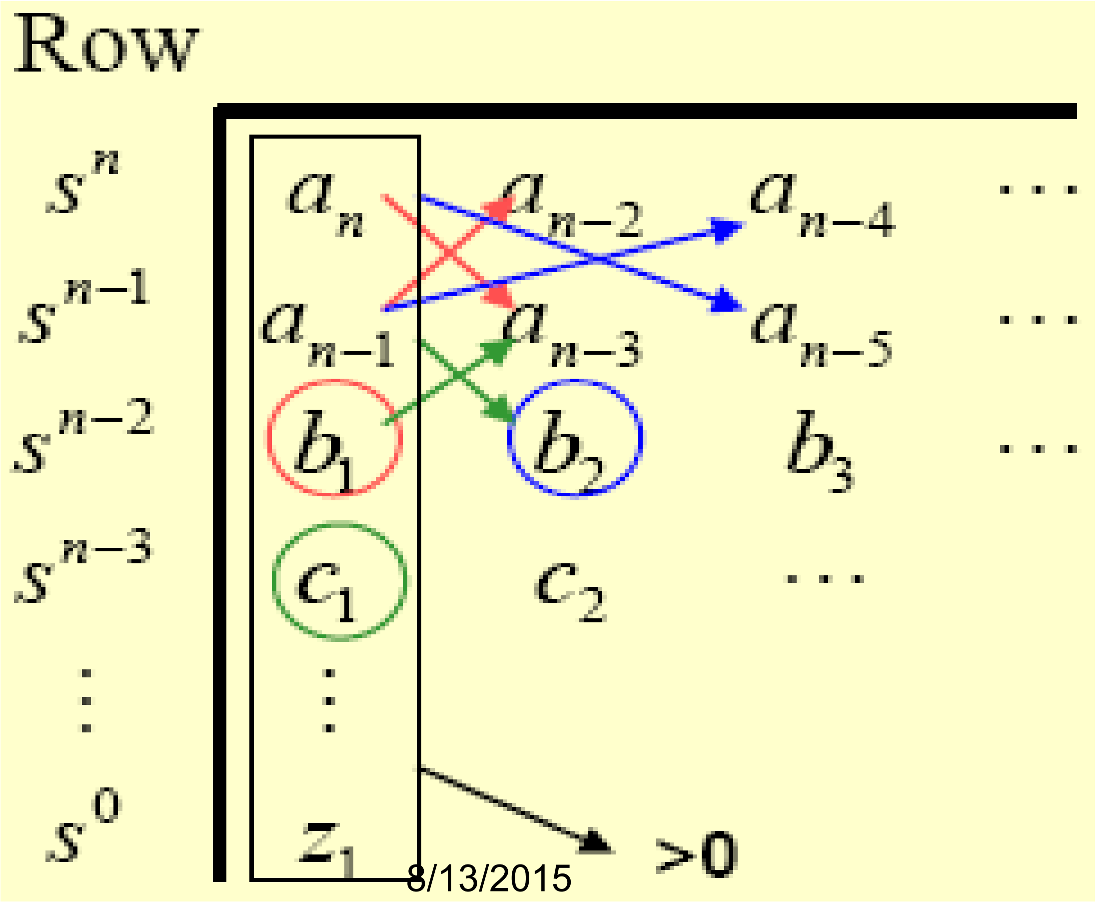
$$\begin{array}{l|lll} s^n & a_n & a_{n-2} & a_{n-4} \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \cdots \end{array}$$

Routhian Rows

- The rows of the schedule are then completed as follows:

s^n	a_n	a_{n-2}	a_{n-4}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
s^0	h_{n-1}		

Routhian Rows



Hurwitz Determinants

- Where,

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix},$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix},$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix},$$

RHC Principle

- The Routh-Hurwitz criterion states that the number of roots of D(s) with positive real parts is equal to the number of changes in sign in the first column of the Routh Array.
- This criterion requires that there be no changes in sign in the first column for a stable system.
- This requirement is both necessary and sufficient.

Four Cases for First Column Array

- No element in the first column is zero.
- There is a zero in the first column, but other elements of the row contain a zero in the first column are non-zero.
- There is a zero in the first column, the other elements of the row contain a zero are also zero.
- There is a zero in the first column, and the other elements of the row contain the zero are also zero, with repeated roots on the jw axis.

Example: Third-Order System

- **The characteristic polynomial of a third-order system is:**

$$s^3 + s^2 + 2s + 24 = 0$$

Example: Third-Order System

- The polynomial satisfies all the necessary conditions because all the coefficients exist and are positive.

- Utilizing the Routh array, we have:

Example: Third-Order System

s^3	1	2
s^2	1	24
s^1	-22	0
s^0	24	0

Routh-Hurwitz Criterion Limitations

- Routh-Hurwitz Criterion is only valid, if the characteristic equation is algebraic with real coefficients. If any one of the coefficients is complex or if the equation is not algebraic, for example, containing experimental functions or sinusoidal functions of s, RHC simply cannot be applied.

Routh-Hurwitz Criterion Limitations

- The RHC cannot be applied to any other stability boundaries in a complex plane, such as the unit circle in the z-plane, which is the stability boundary of discrete data systems.