Root Locus Method

The Root   Locus Method

No matter what we pick K to be, the closed-loop system must always have n poles, where n is the number of poles of G(s).
The root locus must have n branches, each branch starts at a pole of G(s) and goes to a zero of G(s).
If G(s) has more poles than zeros (as is often the case), m < n and we say that G(s) has zeros at infinity. In this case, the limit of G(s) as s -> infinity is zero.

The number of zeros at infinity is n-m, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).

Example

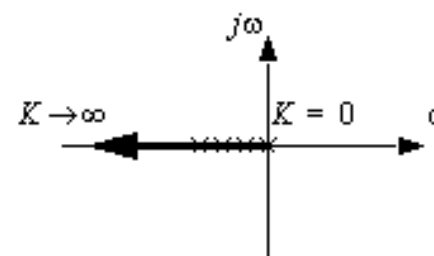$$G(s) = \frac{K}{s} \qquad H(s) = 1$$
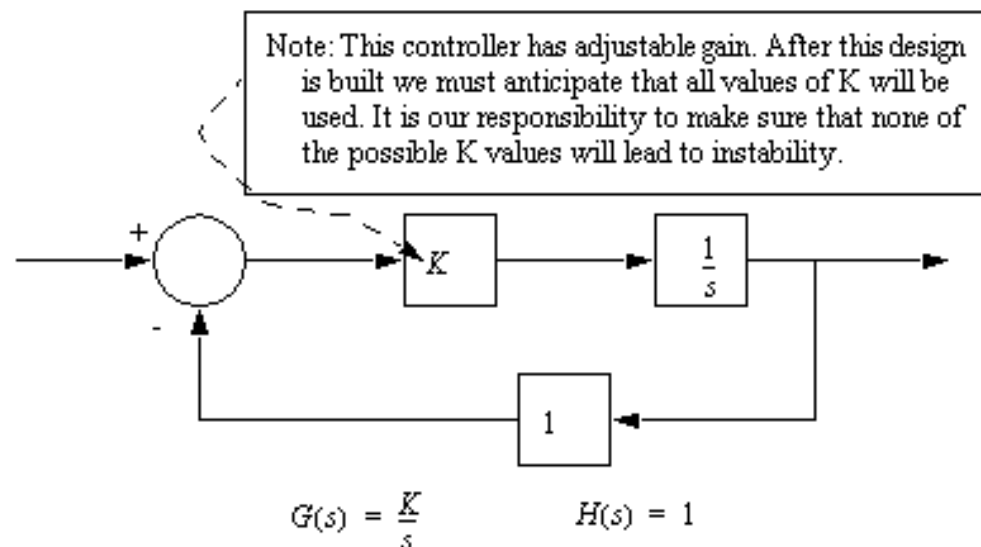
First, we must develop a transfer function for the entire control system.

$$G_S(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.

$s + K = 0$

K	root
0	
1	
2	
3	
etc..	

Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The large the value of K, the more stable the system becomes.

Root Locus – Definition

Root Locus is the method of graphically displaying the roots of a polynomial equation having the following form on the complex plane when the parameter K varies from 0 to ∞:

$$1 + K \cdot G(s) = 0 \quad \text{or} \quad 1 + K \cdot \frac{N(s)}{D(s)} = 0$$

where $N(s)$ and $D(s)$ are *known* polynomials in factorized form:

$$N(s) = (s - z_1)(s - z_2)\cdots \qquad)$$

$$D(s) = (s - p_1)(s - p_2)\cdots \qquad)$$

Conventionally, the N_Z roots of the polynomial $N(s)$, z_1, z_2, ..., z_{Nz}, are called the finite *open-loop zeros*. The N_P roots of the polynomial $D(s)$, p_1, p_2, ..., p_{Np}, are called the finite *open-loop poles*.

Note: By transforming the closed-loop characteristic equation of a feedback controlled system with a single positive design parameter K into the above standard form, one can use the Root Locus technique to determine the range of K that have CL poles in the performance region.

Root Locus Sketching Rules

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \Rightarrow \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_Z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_P})} = 0$$

Rule 1: The number of branches of the root locus is equal to the number of closed-loop poles (or roots of the characteristic equation). In other words, the number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater. $D(s) + KN(s) = 0$

Rule 2: Root locus starts at open-loop poles (*when K*= 0) and ends at open-loop zeros (*when K=∞*). If the number of open-loop poles is greater than the number of open-loop zeros, some branches starting from finite open-loop poles will terminate at zeros at infinity (i.e., go to infinity). If the reverse is true, some branches will start at poles at infinity and terminate at the finite open-loop zeros. $D(s) + KN(s) = 0$ $\boxed{K = 0?}$ $\boxed{K = \infty?}$

Rule 3: Root locus is symmetric about the real axis, which reflects the fact that closed-loop poles appear in complex conjugate pairs.

Rule 4: Along the real axis, the root locus includes all segments that are to the left of an *odd* number of finite <u>real</u> open-loop poles and zeros. $\boxed{\text{Check the phases}}$ $\angle K \dfrac{N(s)}{D(s)} = \angle -1 = \pi [\text{rad}] = 180°$

Root Locus Sketching Rules

Rule 5: If number of poles N_P exceeds the number of zeros N_Z, then as $K \to \infty$, $(N_P - N_Z)$ branches will become asymptotic to straight lines. These straight lines intersect the real axis with angles θ_k at σ_0.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{\text{Sum of open-loop poles} - \text{Sum of open-loop zeros}}{\text{\# of open-loop poles} - \text{\# of open-loop zeros}}$$

$$\theta_k = (2k+1)\frac{\pi}{N_P - N_Z} \text{ [rad]} = (2k+1)\frac{180°}{N_P - N_Z} \text{ [deg]}, \quad k = 0, 1, 2, \cdots$$

If N_Z exceeds N_P, then as $K \to 0$, $(N_Z - N_P)$ branches behave as above.

Rule 6: Breakaway and/or break-in (arrival) points should be the solutions to the following equations:

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \quad \text{or} \quad \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0$$

Root Locus Sketching Rules

Rule 7: The departure angle for a pole p_i (the arrival angle for a zero z_i) can be calculated by slightly modifying the following equation:

angle criterion →

$$\angle(s-z_1)+\angle(s-z_2)+\cdots+\angle(s-z_{N_Z})-\angle(s-p_1)-\angle(s-p_2)-\cdots-\angle(s-p_{N_p})=180°$$

The departure angle q_j from the pole p_j can be calculated by replacing the term $\angle(s-p_j)$ with q_j and replacing all the s's with p_j in the other terms.

Rule 8: If the root locus passes through the imaginary axis (the stability boundary), the crossing point $j\omega$ and the corresponding gain K can be found as follows:

– Replace s in the left side of the closed-loop characteristic equation with $j\omega$ to obtain the real and imaginary parts of the resulting complex number.

– Set the real and imaginary parts to zero, and solve for ω and K. This will tell you at what values of K and at what points on the $j\omega$ axis the roots will cross.

magnitude criterion ⟶

$$K=\frac{|s-p_1||s-p_2|\cdots}{|s-z_1||s-z_2|\cdots}$$

Steps to Sketch Root Locus

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \text{or} \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_Z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_P})} = 0$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i. Mark the open-loop poles and zeros on the complex plane. Use $\times$ to represent open-loop poles and $\circ$ to represent the open-loop zeros.

Step 3: Determine the real axis segments that are on the root locus by applying Rule 4.

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

**Step 6:** (If necessary) Determine the departure and arrival angles using Rule 7.

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Example 1

DC Motor Position Control

In the previous example on the printer paper advance position control, the proportional control block diagram is:
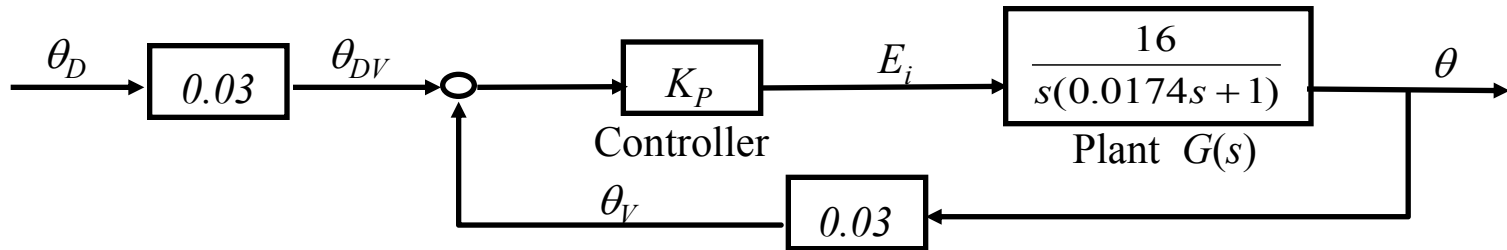
Sketch the root locus of the closed-loop poles as the proportional gain K_P varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1 + K_p G(s) H(s) = 0$$

$$1 + K_p \underbrace{\frac{\overbrace{0.48}^{N(s)}}{s(0.0174s + 1)}}_{D(s)} = 0$$

Example 1

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:
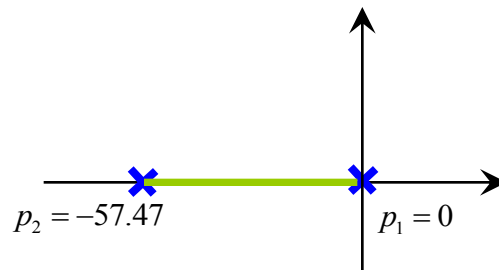
$$1 + \boxed{27.58K_p} \underbrace{\frac{\overbrace{1}^{N(s)}}{s(s+57.47)}}_{D(s)} = 0$$

$$\underbrace{}_{K}$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i:

$\boxed{\text{No open-loop zeros}}$

$\boxed{\text{open-loop poles}}$ $\quad p_1 = 0, \, p_2 = -57.47$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.

Example 1

Step 4: *Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.*

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{-57.47}{2} = -28.74$$

$$\theta_k = (2k+1)\frac{\pi}{N_P - N_Z} \ [\text{rad}] = \begin{cases} \dfrac{\pi}{2} \\ \dfrac{3\pi}{2} \end{cases}$$

Step 5: *(If necessary) Determine the break-away and break-in points using Rule 6.*

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \ \text{ or } \ \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0,$$

$$\frac{d}{ds}\left(\frac{s(0.0174s+1)}{0.48}\right) = 0, 0.0348s + 1 = 0, s = -28.74$$

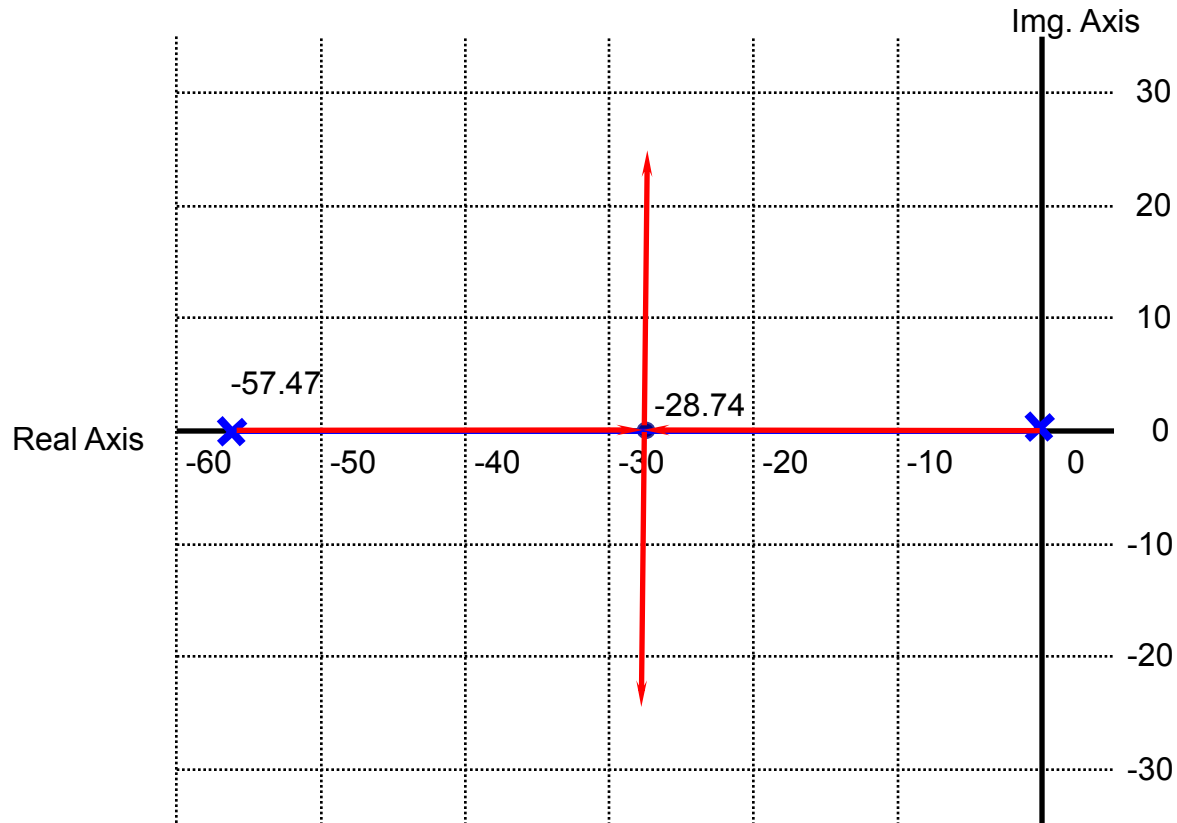*Step 6:* *(If necessary) Determine the departure and arrival angles using Rule 7.*

$$-\angle(p_2 - p_1) - \theta_{p_2} = 180°, \upsilon_{p_2} - \upsilon°$$

$$-\theta_{p_1} - \angle(p_1 - p_2) = 180°, \upsilon_{p_1} - 180°$$

Step 7: *(If necessary) Determine the imaginary axis crossings using Rule 8.*

Could s be pure imaginary in this example?

Example 1
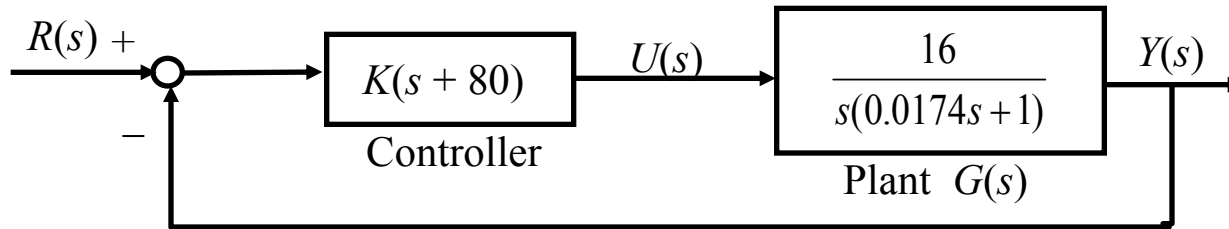
Example 2

A positioning feedback control system is proposed. The corresponding block diagram is:

Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + K(s+80)\frac{16}{s(0.0174s+1)} = 0$$

Example 2

Step 1: *Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:*

$$1 + K \frac{\overbrace{16(s+80)}^{N(s)}}{\underbrace{s(0.0174s+1)}_{D(s)}} = 1 + \boxed{920K} \frac{\overbrace{(s+80)}^{N(s)}}{\underbrace{s(s+57.47)}_{D(s)}} = 0$$

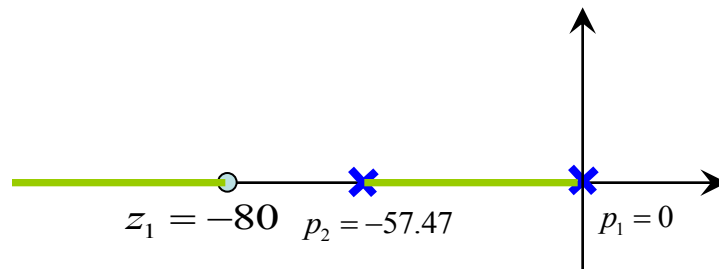*Step 2:* *Find the open-loop zeros, z_i, and the open-loop poles, p_i:*

$$\boxed{\text{open-loop zeros}} \quad z_1 = -80$$

$$\boxed{\text{open-loop poles}} \quad p_1 = 0, \; p_2 = -57.47$$

Step 3: *Determine the real axis segments that are to be included in the root locus by applying Rule 4.*

$z_1 = -80 \quad p_2 = -57.47 \qquad p_1 = 0$

Example 2

Step 4:   Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

Step 5:   (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \ \text{ or } \ \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0,$$

$$\frac{d}{ds}\left(\frac{(s+80)}{s(s+57.47)}\right) = \frac{s(s+57.47)-(s+80)(2s+57.47)}{\left[s(s+57.47)\right]^2} = 0,$$
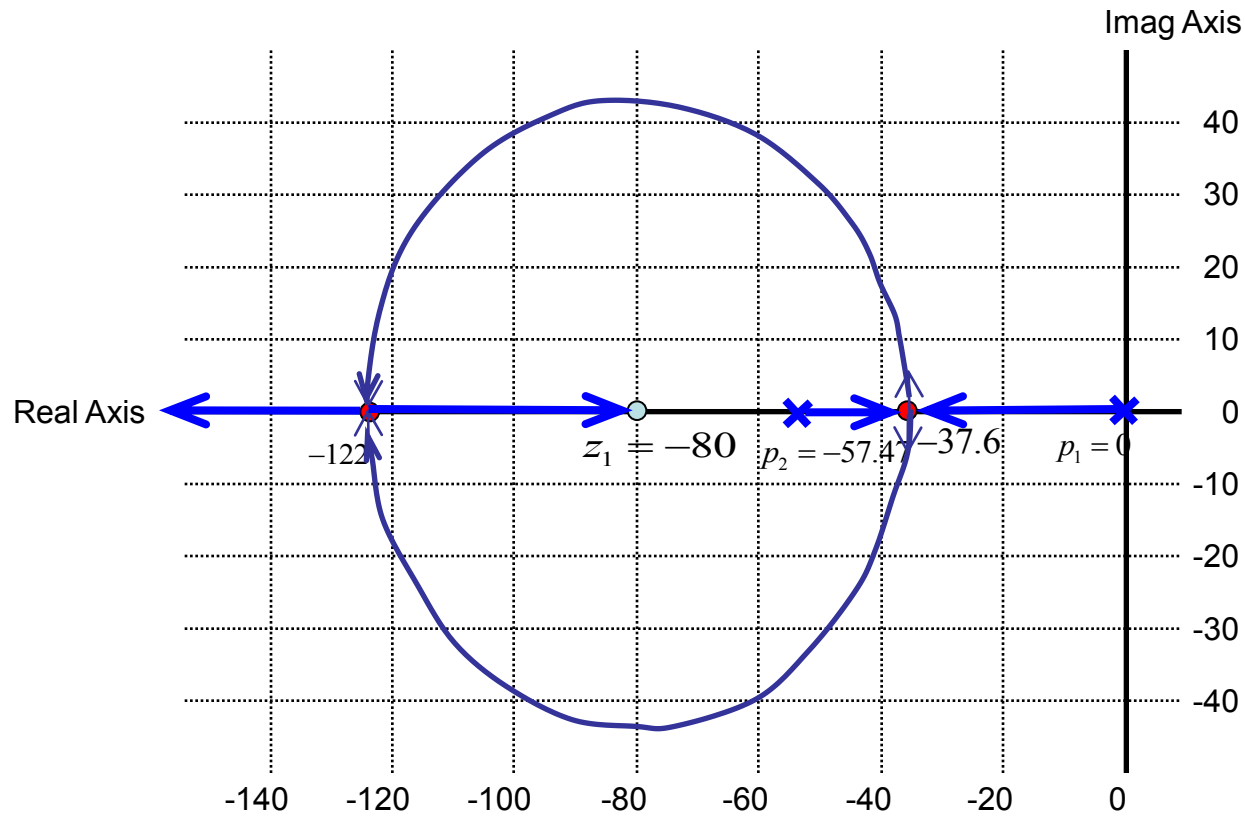
$$s^2 + 160s + 4600 = 0$$

$$s_1 = -122, s_2 = -37.6$$

Example 2

Example 3

A feedback control system is proposed. The corresponding block diagram is:
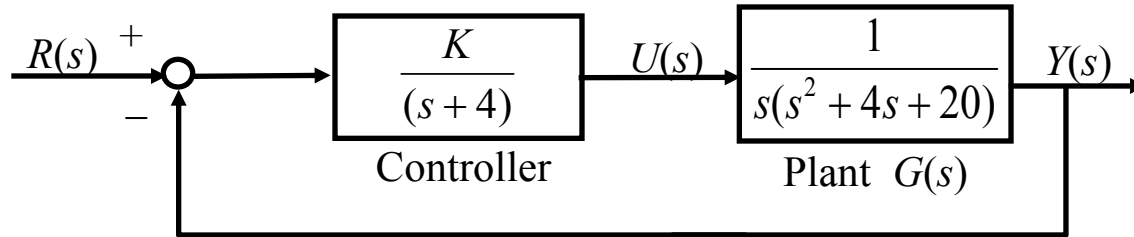
Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + \frac{K}{s+4}\frac{1}{s\left(s^2 + 4s + 20\right)} = 0$$

Example 3

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\overbrace{1}^{N(s)}}{\underbrace{s\left(s^2 + 4s + 20\right)\left(s + 4\right)}_{D(s)}} = 0$$

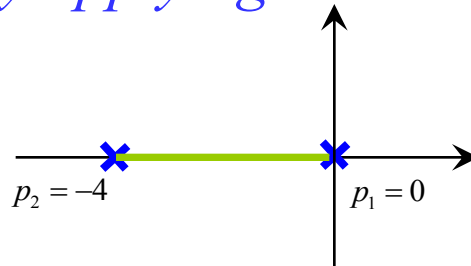*Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i:*

| open-loop zeros | No open-loop zeros |
| open-loop poles | $p_1 = 0, p_2 = -4, p_{3,4} = -2 \pm 4j$ |

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.

$p_2 = -4$ $p_1 = 0$

Example 3

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{0 + (-4) + (-2 + 4j) + (-2 - 4j)}{4 - 0} = -2$$

$$\theta_k = (2k + 1)\frac{\pi}{N_P - N_Z} \; [\text{rad}] = \begin{cases} \dfrac{\pi}{4} \\[4pt] \dfrac{3\pi}{4} \\[4pt] \dfrac{5\pi}{4} \\[4pt] \dfrac{7\pi}{4} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \;\; \text{or} \;\; \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0,$$

$$\frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = \frac{d}{ds}\left(\frac{s\left(s^2 + 4s + 20\right)(s+4)}{1}\right) = \frac{d}{ds}\left(s^4 + 8s^3 + 36s^2 + 80s\right)$$

$$= 4s^3 + 24s^2 + 72s + 80 = 0$$

$$s_1 = -2, s_{2,3} = -2 \pm 2.45j$$

Example 3

Step 6:   (If necessary) Determine the departure and arrival angles using Rule 7.

$$\sum_{i=1}^{N_z} \angle(s - z_i) - \sum_{i=1}^{N_p} \angle(s - p_i) = 180°$$

$p_1 = 0:$ $\theta_{p_1} = 180°$ $p_3 = -2 + 4j:$ $\theta_{p_3} = -90°$

$p_2 = -4:$ $\theta_{p_2} = 0°$ $p_4 = -2 - 4j:$ $\theta_{p_4} = 90°$

Step 7:   (If necessary) Determine the imaginary axis crossings using Rule 8.

$$1 + K \frac{1}{s\left(s^2 + 4s + 20\right)(s + 4)} = 0 \Rightarrow \boxed{\begin{array}{l} s\left(s^2 + 4s + 20\right)(s + 4) + K = 0 \\[2mm] \Leftrightarrow s^4 + 8s^3 + 36s^2 + 80s + K = 0 \end{array}}$$ CLCE
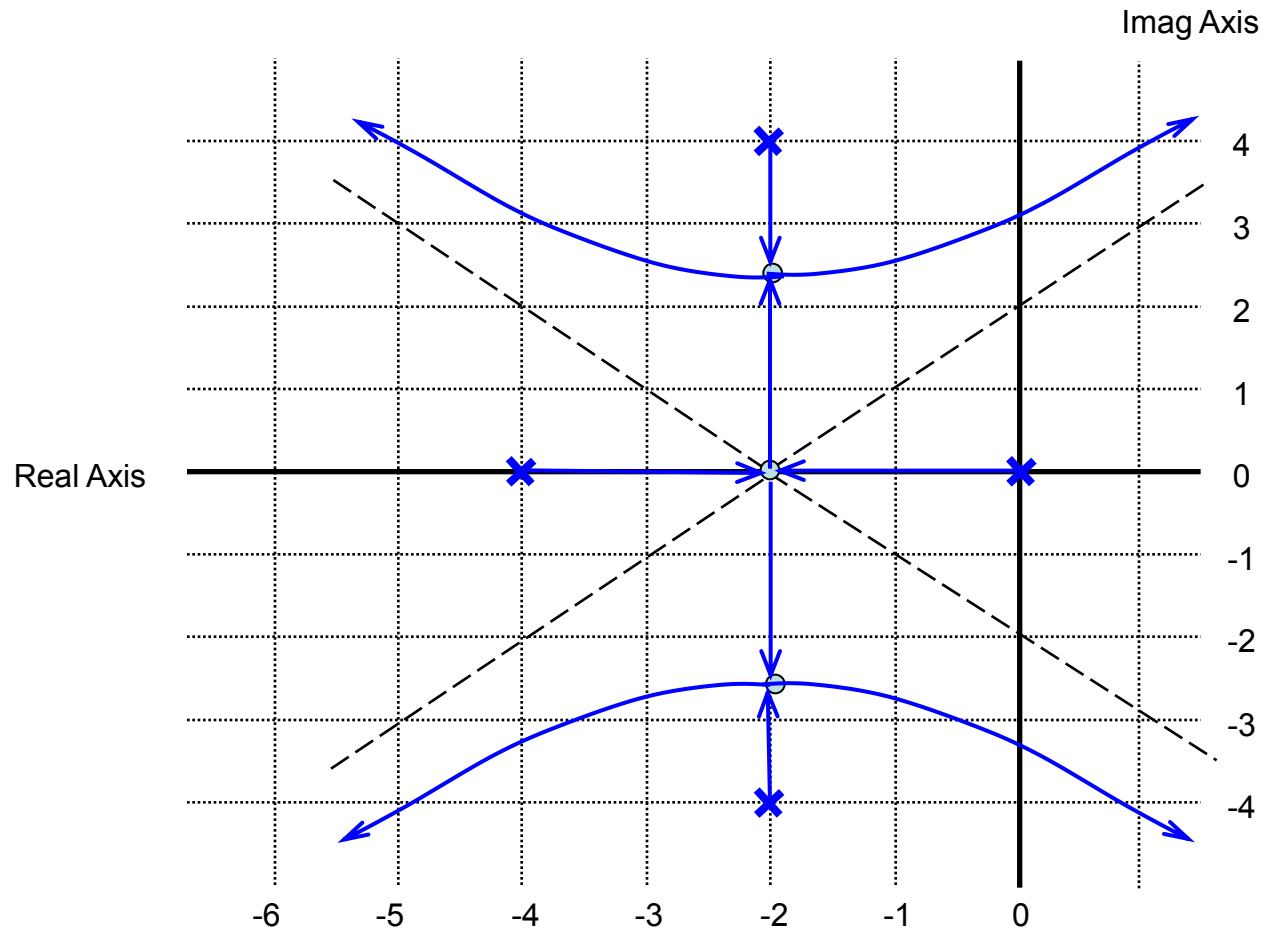
$$\xRightarrow{s = j\omega} \left(\omega^4 - 36\omega^2 + K\right) + \left(-8\omega^3 + 80\omega\right)j = 0$$

$$\Rightarrow \begin{cases} \omega^4 - 36\omega^2 + K = 0 \\ -8\omega^3 + 80\omega = 0 \end{cases} \Rightarrow \begin{cases} K_1 = 0 \\ \omega_1 = 0 \end{cases}, \begin{cases} K_2 = 260 \\ \omega_2 = \sqrt{10} = 3.16 \end{cases}$$
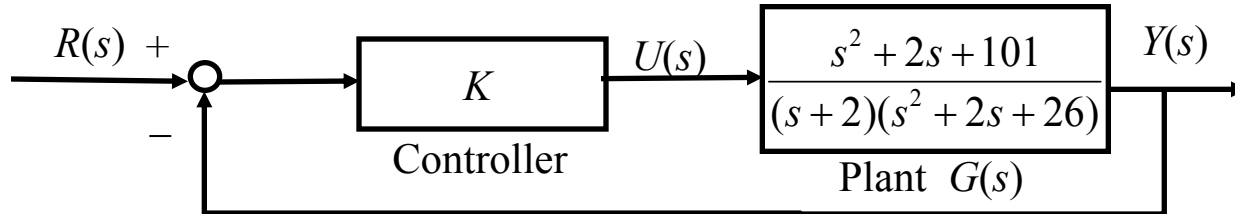
Example 3

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Example 4

A feedback control system is proposed. The corresponding block diagram is:

Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1 + K \frac{s^2 + 2s + 101}{(s+2)(s^2 + 2s + 26)} = 0$$

Example 4

Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\overbrace{s^2 + 2s + 101}^{N(s)}}{\underbrace{(s+2)(s^2 + 2s + 26)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i:

open-loop zeros $s^2 + 2s + 101 = (s+1)^2 + 100 = 0, z_{1,2} = -1 \pm 10j$

open-loop poles $(s+2)((s+1)^2 + 25) = 0, p_1 = -2, p_{2,3} = -1 \pm 5j$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.

$p_1 = -2$
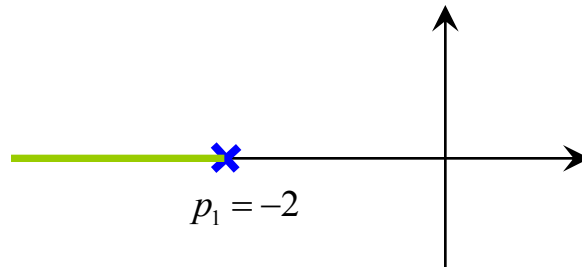
Example 4

Step 4: *Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.*

$$N_p - N_z = 1 \qquad \text{One asymptote} \qquad \theta_k = (2k+1) \times 180° - 180°$$

Step 5: *(If necessary) Determine the break-away and break-in points using Rule 6.*

Step 6: *(If necessary) Determine the departure and arrival angles using Rule 7.*

$$p_1 = -2 \qquad \theta_{p_1} = 180°$$

$$z_1 = -1 + 10j \quad \theta_{z_1} + 90° - \tan^{-1}(10) - 90° - 90° = 180°$$

$$\theta_{z_1} = 354° = -6°$$

$$p_2 = -1 + 5j \qquad \theta_{p_2} = 11°$$

$$z_2 = -1 - 10j \quad \theta_{z_2} = 6° \qquad p_3 = -1 - 5j \qquad \theta_{p_2} = -11°$$

Step 7: *(If necessary) Determine the imaginary axis crossings using Rule 8.*

$$(s+2)(s^2 + 2s + 26) + K(s^2 + 2s + 101) = 0$$

$$\Leftrightarrow s^3 + (4+K)s^2 + (30+2K)s + (52+101K) = 0$$

$$\overset{s=j\omega}{\Rightarrow} \left[(52+101K) - (4+K)\omega^2\right] + \left[(30+2K) - \omega^2\right]\omega j = 0$$

$$\begin{cases} (52+101K) - (4+K)\omega^2 = 0 \\ \left[(30+2K) - \omega^2\right]\omega = 0 \end{cases} \Rightarrow \begin{cases} \omega_1 = 0 \\ K_1 = -\dfrac{52}{101} \end{cases}, \begin{cases} \omega_2 = 9.5 \\ K_2 = 30.4 \end{cases}, \begin{cases} \omega_3 = 5.7 \\ K_3 = 1.1 \end{cases}$$

Example 4

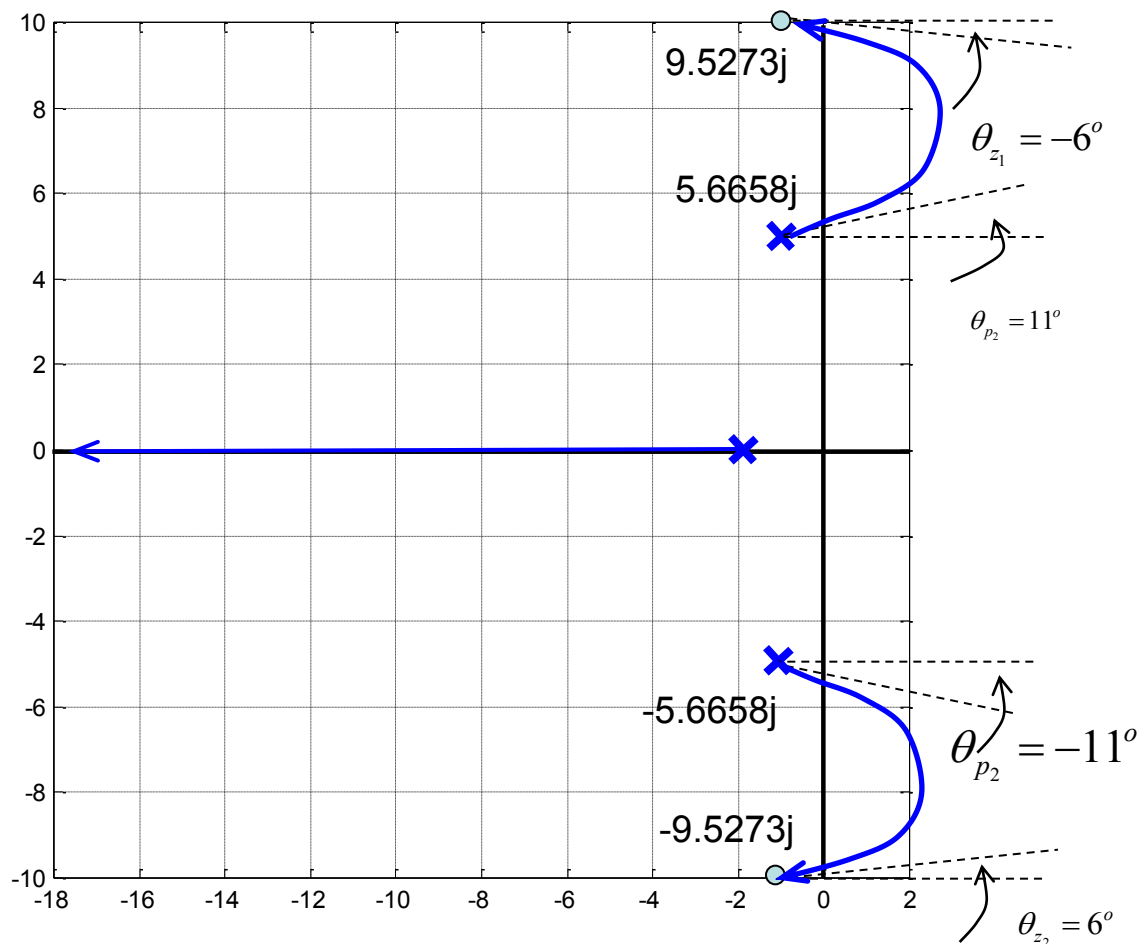*Step 8:   Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.*

Stability condition

$$0 < K < 1.1$$

or

$$K > 30.4$$

9.5273j

5.6658j

$\theta_{z_1} = -6^o$

$\theta_{p_2} = 11^o$

-5.6658j

-9.5273j

$\theta_{p_2}' = -11^o$

$\theta_{z_2} = 6^o$

Root Locus as an Analysis/Design Tool

Mechanical system response depends on the location of the system characteristic values, i.e., poles of the system transfer function. Since root locus tells us how the system poles vary w.r.t. a parameter *K,* we can use root locus to analyze the effect of parameter variation on system performance.

Ex: (*Motion Control of Hydraulic Cylinders*)

Recall the example of the flow control of a hydraulic cylinder that takes into account the capacitance effect of the pressure chamber. The plant transfer function is:

$$G(s) = \frac{V(s)}{Q_{IN}(s)} = \frac{A}{MCs^2 + BCs + A^2}$$

where *M* is the mass of the load; *C* is the flow capacitance of the pressure chamber; *A* is the effective area of the piston and *B* is the viscous friction coefficient.

Q: *How would the plant parameters affect the system response* ?
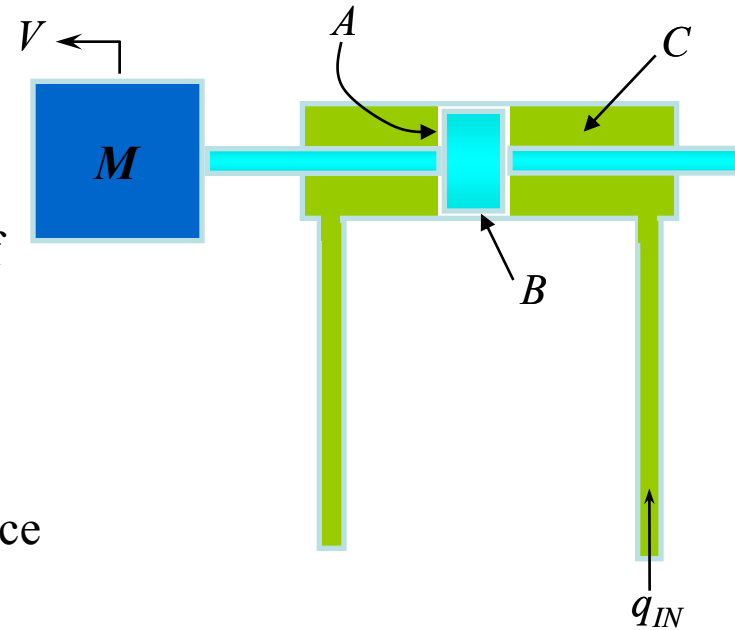
Root Locus as an Analysis/Design Tool

- **Effect of load (M) on system performance:**

  System characteristic equation:

  $$MCs^2 + BCs + A^2 = 0$$

  Transform characteristic equation into standard form for root locus analysis by identifying the parameter that is to be varied. In this case, the load mass M is the varying parameter:

  Standard form $\qquad 1 + K\dfrac{N(s)}{D(s)} = 0 \qquad 1 + M\dfrac{C}{BC}\dfrac{\overset{s^2}{\overbrace{N(s)}}}{\underbrace{s + \dfrac{A^2}{BC}}_{D(s)}} = 0$
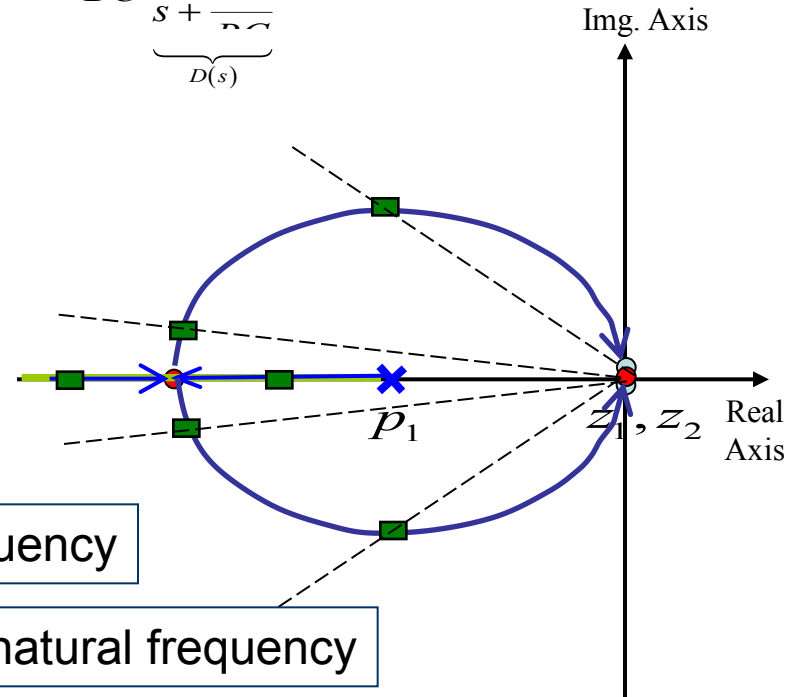
  Varying parameter

  open-loop zeros $\qquad z_1 = z_2 = 0$

  open-loop poles $\qquad p_1 = -\dfrac{A^2}{BC}$

  $$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \Rightarrow s = -\frac{2A^2}{BC}, s = 0$$

  Small M: less overshoot and high natural frequency

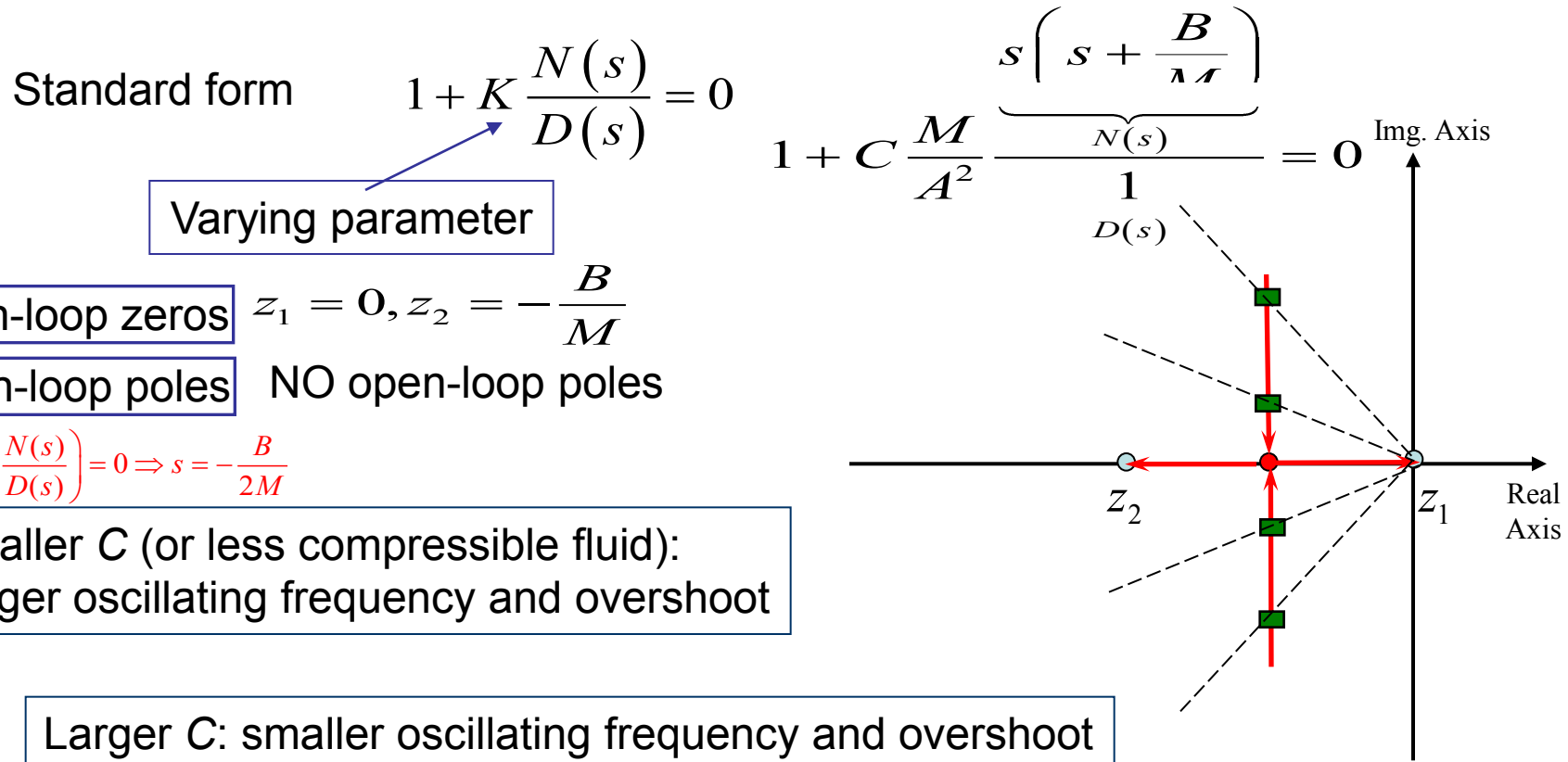 As M increases: larger overshoot and lower natural frequency

  Think about the settling time

Root Locus as an Analysis/Design Tool

- **Effect of flow capacitance (C) on system performance:**

System characteristic equation: $MCs^2 + BCs + A^2 = 0$

Transform characteristic equation into standard form for root locus analysis by identifying the parameter that is to be varied. In this case, the flow capacitance C is the varying parameter:

Standard form $\qquad 1 + K\dfrac{N(s)}{D(s)} = 0$

Varying parameter

$$1 + C\frac{M}{A^2}\frac{\overbrace{s\left(s + \dfrac{B}{M}\right)}^{N(s)}}{\underbrace{1}_{D(s)}} = 0$$

open-loop zeros $\quad z_1 = 0,\ z_2 = -\dfrac{B}{M}$

open-loop poles $\quad$ NO open-loop poles

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \Rightarrow s = -\frac{B}{2M}$$

Smaller C (or less compressible fluid):
Larger oscillating frequency and overshoot

Larger C: smaller oscillating frequency and overshoot

Img. Axis

z_2

z_1

Real Axis

Root Locus as an Analysis/Design Tool

- **Effect of friction (*B*) on system performance:**

System characteristic equation:
$$MCs^2 + BCs + A^2 = 0$$

Transform characteristic equation into standard form for root locus analysis by identifying the parameter that is to be varied. In this case, the viscous friction coefficient B is the varying parameter:

Standard form
$$1 + K\frac{N(s)}{D(s)} = 0 \qquad 1 + B\frac{C}{MC}\frac{\overbrace{N(s)}^{s}}{\underbrace{s^2 + \frac{A^2}{MC}}_{D(s)}} = 0$$

Varying parameter

open-loop zeros

open-loop poles

$$z_1 = 0$$
$$p_{1,2} = \pm j\sqrt{\frac{A^2}{MC}}$$

$$\frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0 \Rightarrow s = -\sqrt{\frac{A^2}{MC}}$$

Smaller *B*:
Larger oscillating frequency and overshoot

Larger *B*: smaller oscillating frequency and overshoot

settling time?