7.1 INTERRELATION BETWEEN TIME AND FREQUENCY RESPONSE

The frequency domain and time-domain performances of a linear system are correlated such that the time-domain properties of the system can be predicted on the basis of frequency-domain characteristics.

For example, transient response can be obtained from frequency response through Fourier Integral. In this case, a higher order system is approximated as a second-order system to simplify the analysis.

Transfer function is a complex function of frequency having magnitude and phase angle. These can be represented by a graphical plot. For a linear SISO system with a sinusoidal input, $r(t) = A \sin(\omega t)$, the steady-state output will also be a sinusoidal of the same frequency with different amplitude and phase which may be expressed as $y(t) = B \sin(\omega t + \varphi)$.

Let the transfer functi

where ω_n = natural frequ

Frequency response is [illegible] $j\omega$ in the system transfer function as

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \qquad \ldots(7.1)$$

Normalizing the driving signal frequency with $\mu = \omega/\omega_n$, we have

$$T(j\mu) = \frac{1}{(1-\mu^2) + j2\zeta\mu} \qquad \ldots(7.2)$$

From Eq. (7.1) we can write the magnitude and phase response as follows:

$$|M(j\omega)| = \frac{1}{\sqrt{(1-\mu^2)^2 + (2\zeta\mu)^2}} \qquad \ldots(7.3)$$

$$\angle M(j\omega) = \varphi = -\tan^{-1}\left[\frac{2\zeta\mu}{(1-\mu^2)}\right] \qquad \ldots(7.4)$$

The steady-state output of the system for sinusoidal input of unit magnitude and variable frequency ω is

$$y(t) = \frac{1}{\sqrt{(1-\mu^2)^2 + (2\zeta\mu)^2}} \sin\left[\omega t - \tan^{-1}\left[\frac{2\zeta\mu}{(1-\mu^2)}\right]\right] \qquad \ldots(7.5)$$

From Eqs (7.3) and (7.4), we obtain the normalized frequency μ,

$\mu = 0$	$\mu = 1$	$\mu = \infty$
$M = 1$ and $\varphi = 0$	$M = 1/2\zeta$ and $\varphi = -\pi/2$	$M \to 0$ and $\varphi \to -\pi$

The magnitude and phase plot for normalized frequency μ for a certain value of ζ is shown in Fig. 7.1(a) and 7.1 (b).

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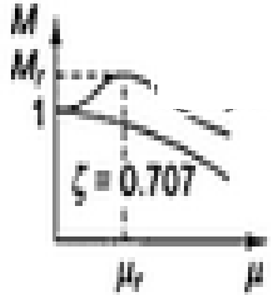

Fig. 7.1(a) *Magnitude response of second-order system for normalized frequency 'μ'*

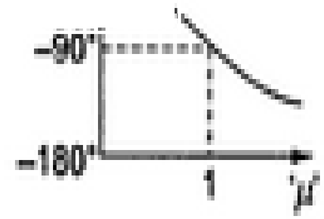

Fig. 7.1(b) *Phase response of a second-order system for system for normalized frequency 'μ'*

The frequency for which M has a peak value is known as resonant frequency ω_r. At this frequency, the slope of the magnitude curve is zero.

with $\mu_r = (\omega_r / \omega_n)$, defined as the normalized resonant frequency, $\frac{dM}{d\mu}|_{\mu=\mu_r} = 0$ gives

$$4\mu_r^3 - 4\mu_r + 8\zeta^2\mu_r = 0$$

or

$$\mu_r = \sqrt{(1 - 2\zeta^2)}$$

i.e., $$\omega_r = \omega_n\sqrt{(1-2\zeta^2)} \qquad \ldots(7.6)$$

From Eq. (7.3), the maximum value of magnitude known as resonant peak is given by

$$\ldots(7.7)$$

and phase angle φ of $M($

$$\angle M(j\omega) = \varphi_r = -\tan^{-1}\left[\frac{\sqrt{(1-2\zeta^2)}}{\zeta}\right] \qquad \ldots(7.8)$$

In Chapter 3 for the step response, the maximum overshoot M_p and damped frequency of oscillation ω_d (with damping ratio $0 < \zeta \le 1$) have been found as

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad \ldots(7.9)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \ldots(7.10)$$

From Eqs (7.6) and (7.10), we observe that ω_r and ω_d are related as follows:

$$\omega_r = \omega_d\,\frac{\sqrt{1-2\zeta^2}}{\sqrt{1-\zeta^2}} \qquad \ldots(7.11)$$

The following observations may be made from equations (7.6), (7.7), (7.9) and (7.11).

- For small values of ζ, the damped frequency ω_d and resonant frequency ω_r are nearly equal, and for large value of ω_r, the time response is faster.
- Both M_r and M_p, being functions of ζ, decrease as ζ is increased.
- As $\zeta \to 0$, $\omega_r \to \omega_n$, and $M_r \to \infty$
- For $0 < \zeta \le 1/\sqrt{2}$, ω_r is always less than ω_n and resonant peak has a value greater than 1 and for $\zeta > 1/\sqrt{2}$, the magnitude M decreases monotonically from $M = 1$ at $\mu = 0$ with increasing μ.

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==> for $\zeta > 1/\sqrt{2}$ there is no resonant peak and the greatest value of M equals 1.

Thus, for a second-order system, M_r is indicative of its damping factor ζ for $0 < \zeta \le 1/\sqrt{2}$ and ω_r is indicative of its natural frequency for a given value of ζ, and hence is indicative of its speed of response (as $t_s = 4/\zeta\,\omega_n$).

Therefore, M_r and ω_r of frequency response could be used as performance indices for a second-order system.

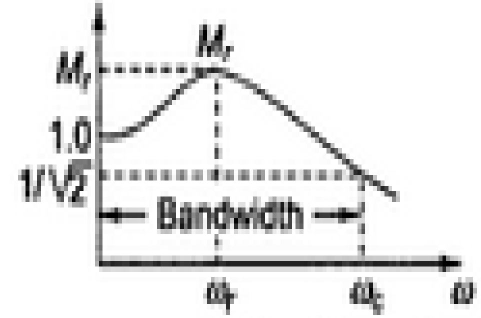

Fig. 7.2 *Bandwidth of a second-order system*

The range of frequency over which M is equal to or greater than $1/\sqrt{2}$ is known as bandwidth ω_b and at the cutoff frequency ω_c, $M = 1/\sqrt{2}$.

Bandwidth indicates noise filtering characteristics of the system. The expression for bandwidth is obtained using Eq. (7.1) as follows

At the normalized bandwidth $\mu_b = \omega_b/\omega_n$

$$M = \frac{1}{\sqrt{(1-\mu_b^2)^2 + (2\zeta\,\mu_b)^2}} = \frac{1}{\sqrt{2}}$$

and after simplifying it, we obtain

$$\omega_b = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

Example 7.1 Let us consider a unity feedback control system as

$$G(s) = \frac{81}{s(s+8)}$$

Evaluate resonance peak and resonant frequency of the closed-loop system.

First of all, calculate closed-loop transfer function of the above system considering $H(s) = 1$

$$\frac{Y(s)}{X(s)} = \frac{81}{s^2 + 8s + 81}$$

Comparing it with a standard second-order system , we get $\omega_n = \sqrt{81} = 9$ and $\zeta = 4/9$.

And using Eqs (7.6) and (7.7)

$$\omega_r = \omega_n \sqrt{(1 - 2\zeta^2)} = 7 \text{ rad/sec}$$

$$M_r = 1 / 2\zeta \sqrt{(1 - \zeta^2)} = 1.26$$