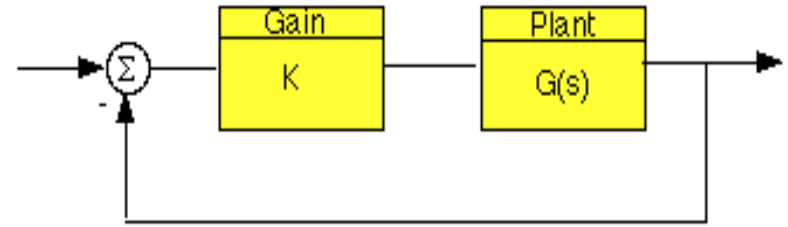


Gain and Phase Margin

Let's say that we have the following system:



where K is a variable (constant) gain and $G(s)$ is the plant under consideration.

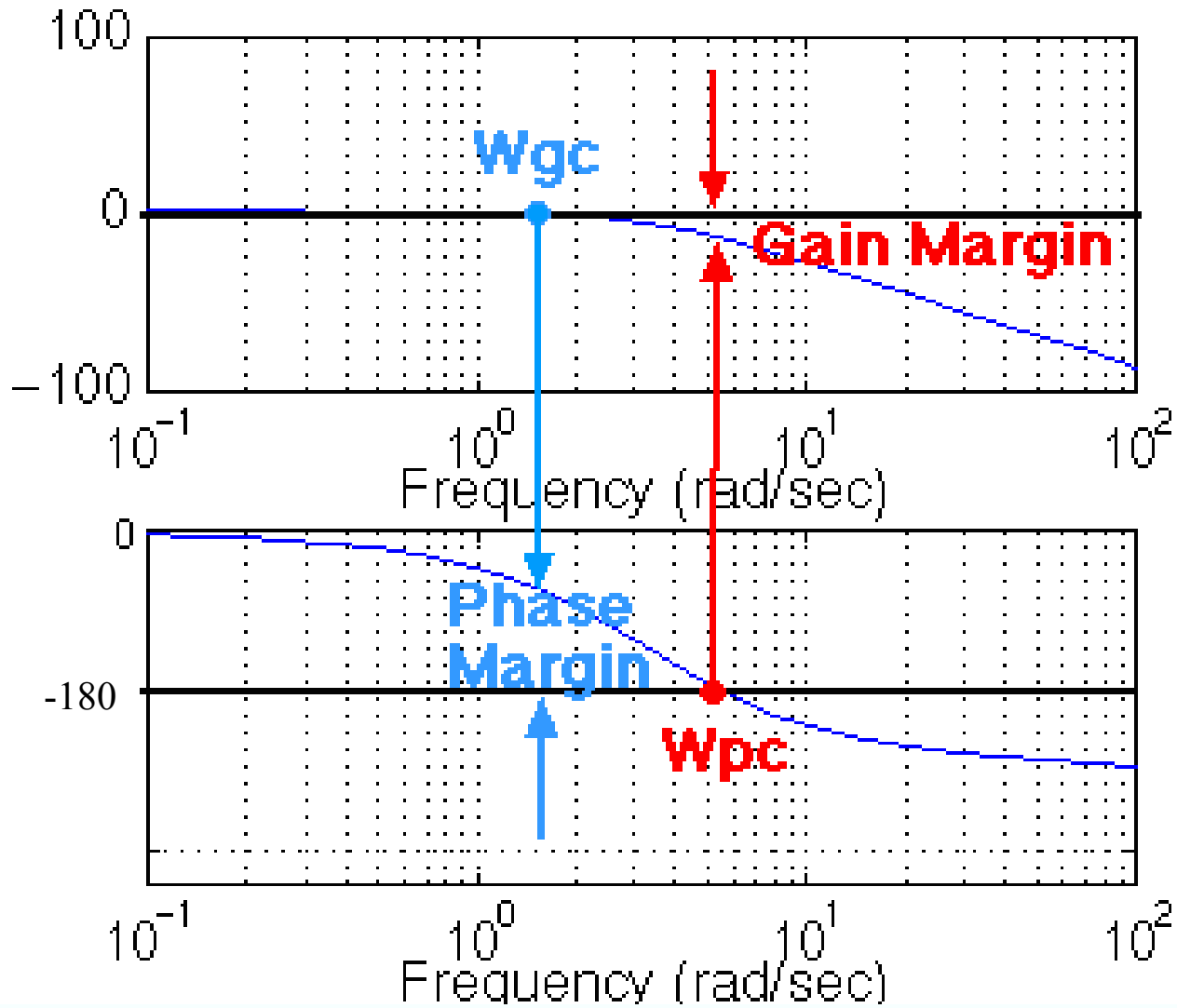
The gain margin is defined as the change in open loop gain required to make the system unstable. Systems with greater gain margins can withstand greater changes in system parameters before becoming unstable in closed loop. Keep in mind that unity gain in magnitude is equal to a gain of zero in dB.

The phase margin is defined as the change in open loop phase shift required to make a closed loop system unstable.

The phase margin is the difference in phase between the phase curve and -180 deg at the point corresponding to the frequency that gives us a gain of 0 dB (the gain cross over frequency, ω_{gc}).

Likewise, the gain margin is the difference between the magnitude curve and 0 dB at the point corresponding to the frequency that gives us a phase of -180 deg (the phase cross over frequency, ω_{pc}).

Gain and Phase Margin

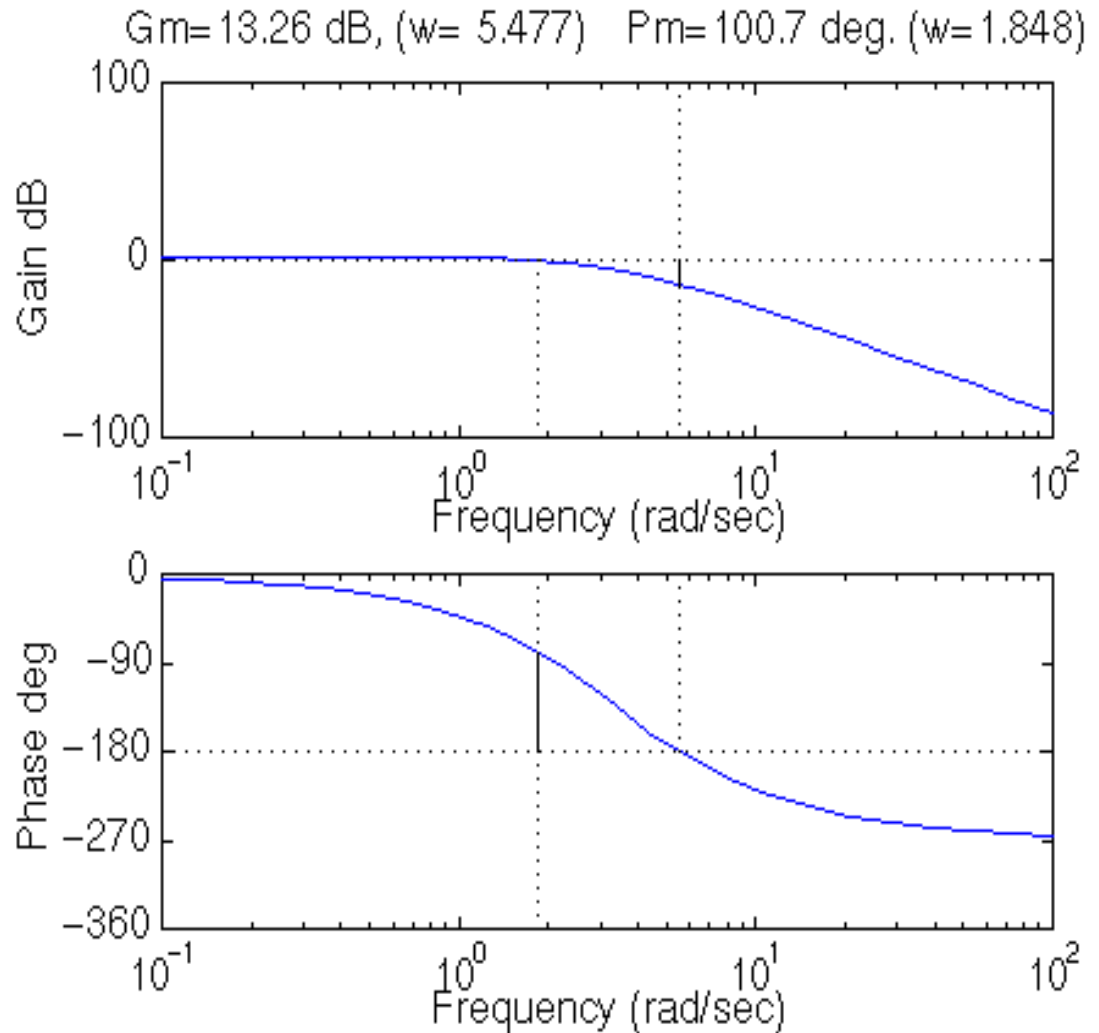


Gain and Phase Margin

We can find the gain and phase margins for a system directly, by using MATLAB. Just enter the `margin` command.

This command returns the gain and phase margins, the gain and phase cross over frequencies, and a graphical representation of these on the Bode plot.

```
margin(50,[1 9 30 40])
```



Gain and Phase Margin

Magnitude:

$$\text{db}(G, \omega) := 20 \cdot \log(|G(j \cdot \omega)|)$$

Phase shift:

$$\text{ps}(G, \omega) := \frac{180}{\pi} \cdot \arg(G(j \cdot \omega)) - 360 \cdot (\text{if}(\arg(G(j \cdot \omega)) \geq 0, 1, 0))$$

Assume

$$K := 2 \quad G(s) := \frac{K}{s \cdot (1 + s) \cdot \left(1 + \frac{s}{3}\right)}$$

Next, choose a frequency range for the plots (use powers of 10 for convenient plotting):

$$\text{lowest frequency (in Hz):} \quad \omega_{\text{start}} := .01 \quad \text{number of points:} \quad N := 50$$

$$\text{highest frequency (in Hz):} \quad \omega_{\text{end}} := 100$$

$$\text{step size:} \quad r := \log\left(\frac{\omega_{\text{start}}}{\omega_{\text{end}}}\right) \cdot \frac{1}{N}$$

$$\text{range for plot:} \quad i := 0..N \quad \text{range variable:} \quad \omega_i := \omega_{\text{end}} \cdot 10^{i \cdot r} \quad s_i := j \cdot \omega_i$$

Gain and Phase Margin

Guess for **crossover frequency** $\omega_c := 1$

Solve for the gain crossover frequency:

$$\omega_c := \text{root}(\text{db}(G, \omega_c), \omega_c) \quad \omega_c = 1.193$$

Calculate the **phase margin**

$$\text{pm} := \text{ps}(G, \omega_c) + 180 \quad \text{pm} = 18.265 \text{ degrees}$$

Gain Margin

Now using the phase angle plot, estimate the frequency at which the phase shift crosses 180 deg

$$\omega_{\text{gm}} := 1.8$$

Solve for ω at the phase shift point of 180 degrees:

$$\omega_{\text{gm}} := \text{root}(\text{ps}(G, \omega_{\text{gm}}) + 180, \omega_{\text{gm}})$$

$$\omega_{\text{gm}} = 1.732$$

Calculate the **gain margin**

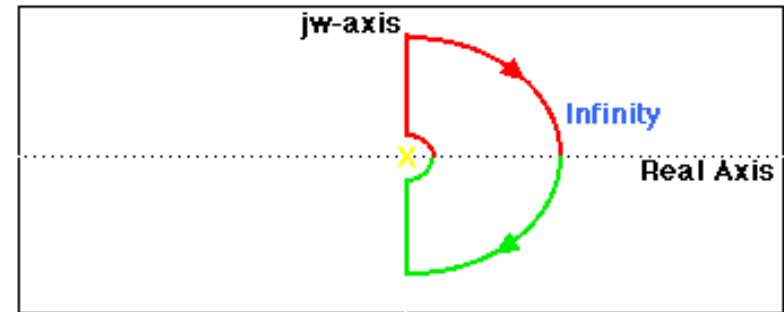
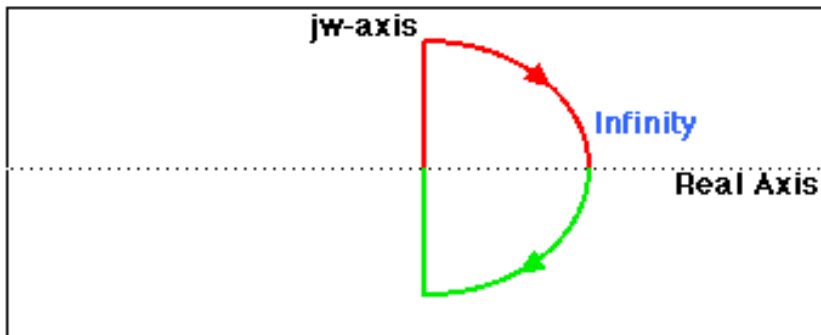
$$\text{gm} := -\text{db}(G, \omega_{\text{gm}}) \quad \text{gm} = 6.021$$

The Nyquist Stability Criterion

The Nyquist plot allows us also to predict the stability and performance of a closed-loop system by observing its open-loop behavior. The Nyquist criterion can be used for design purposes regardless of open-loop stability (Bode design methods assume that the system is stable in open loop). Therefore, we use this criterion to determine closed-loop stability when the Bode plots display confusing information.

The Nyquist diagram is basically a plot of $G(j\omega)$ where $G(s)$ is the open-loop transfer function and ω is a vector of frequencies which encloses the entire right-half plane. In drawing the Nyquist diagram, both positive and negative frequencies (from zero to infinity) are taken into account. In the illustration below we represent positive frequencies in red and negative frequencies in green. The frequency vector used in plotting the Nyquist diagram usually looks like this (if you can imagine the plot stretching out to infinity):

However, if we have open-loop poles or zeros on the $j\omega$ axis, $G(s)$ will not be defined at those points, and we must loop around them when we are plotting the contour. Such a contour would look as follows:



The Cauchy criterion

The Cauchy criterion (from complex analysis) states that when taking a closed contour in the complex plane, and mapping it through a complex function $G(s)$, the number of times that the plot of $G(s)$ encircles the origin is equal to the number of zeros of $G(s)$ enclosed by the frequency contour minus the number of poles of $G(s)$ enclosed by the frequency contour. Encirclements of the origin are counted as positive if they are in the same direction as the original closed contour or negative if they are in the opposite direction.

When studying feedback controls, we are not as interested in $G(s)$ as in the closed-loop transfer function:

$$\frac{G(s)}{1 + G(s)}$$

If $1 + G(s)$ encircles the origin, then $G(s)$ will enclose the point -1 .

Since we are interested in the closed-loop stability, we want to know if there are any closed-loop poles (zeros of $1 + G(s)$) in the right-half plane.

Therefore, the behavior of the Nyquist diagram around the -1 point in the real axis is very important; however, the axis on the standard nyquist diagram might make it hard to see what's happening around this point

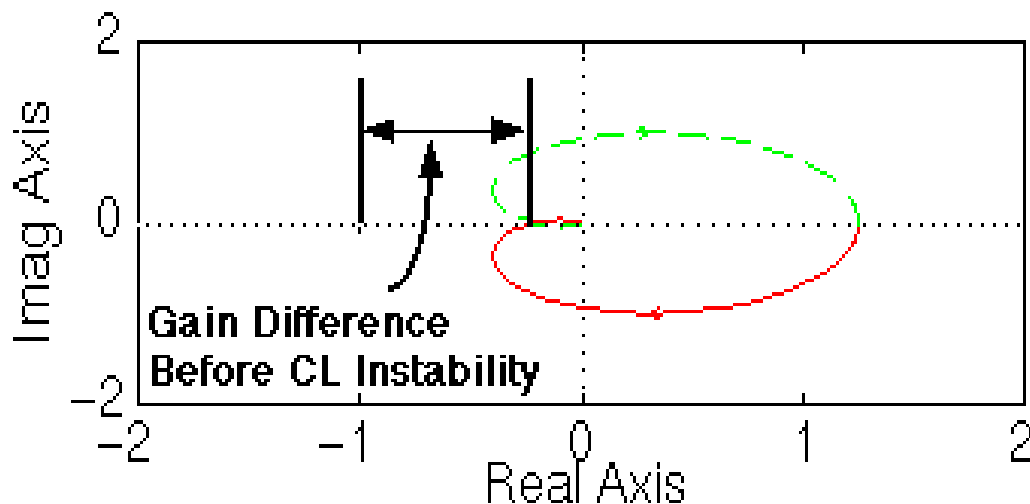
Gain and Phase Margin

Gain Margin is defined as the change in open-loop gain expressed in decibels (dB), required at 180 degrees of phase shift to make the system unstable. First of all, let's say that we have a system that is stable if there are no Nyquist encirclements of -1, such as :

$$\frac{50}{s^3 + 9s^2 + 30s + 40}$$

Looking at the roots, we find that we have no open loop poles in the right half plane and therefore no closed-loop poles in the right half plane if there are no Nyquist encirclements of -1. Now, how much can we vary the gain before this system becomes unstable in closed loop?

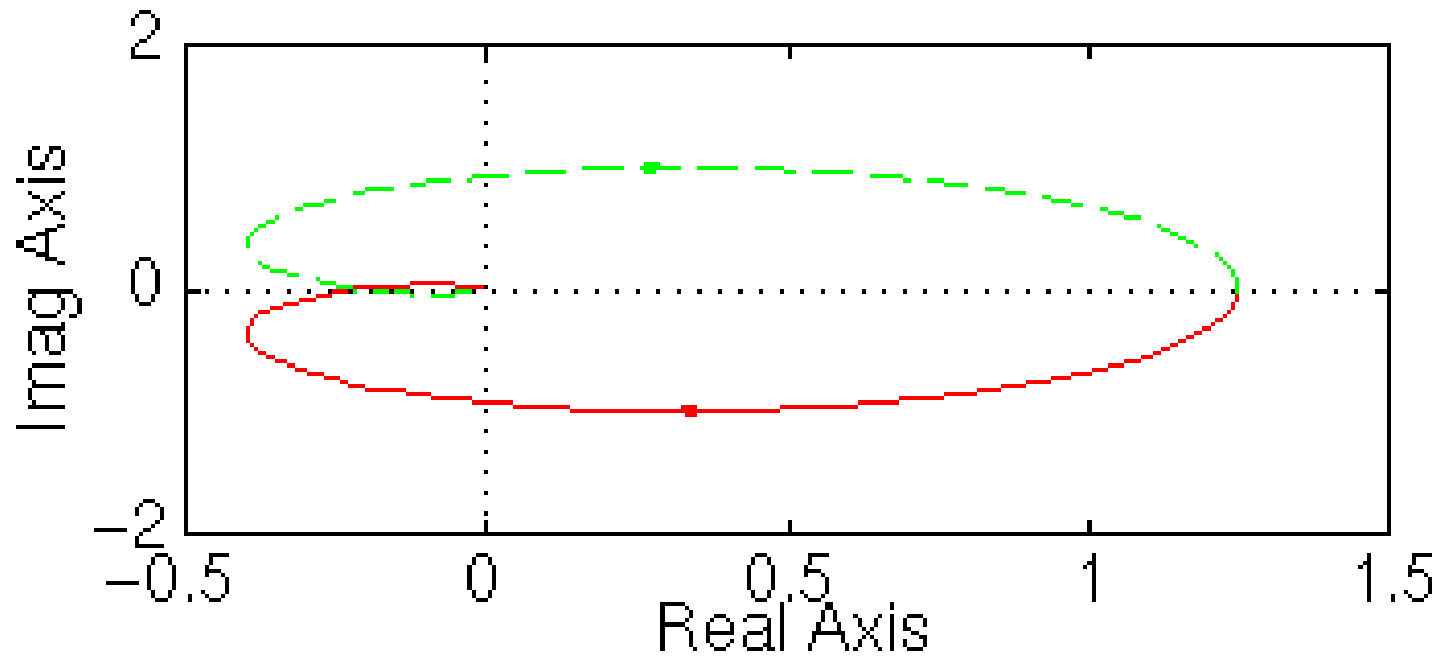
The open-loop system represented by this plot will become unstable in closed loop if the gain is increased past a certain boundary.



The Nyquist Stability Criterion

and that the Nyquist diagram can be viewed by typing:

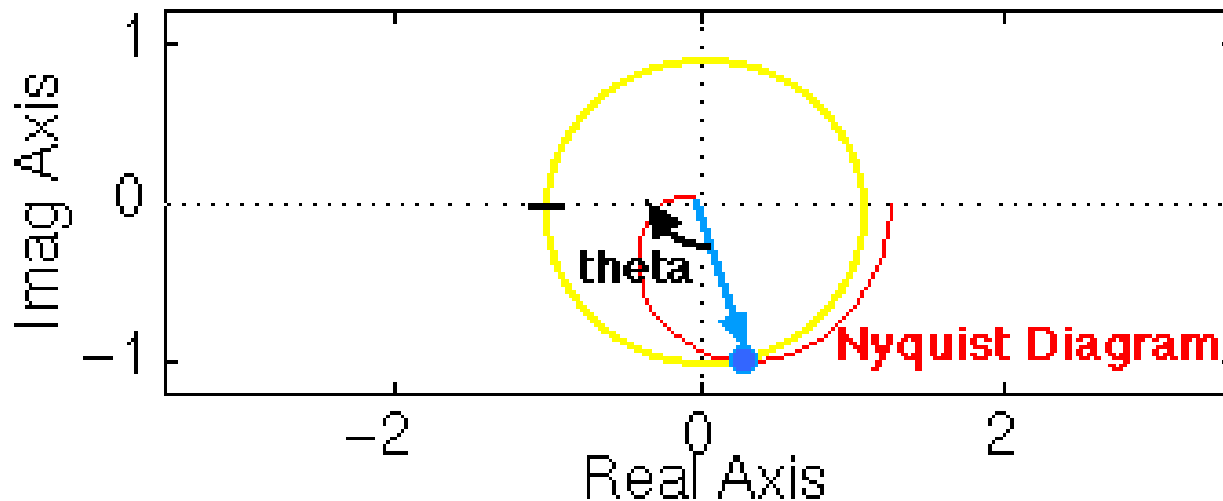
```
nyquist (50, [1 9 30 40 ])
```



Gain and Phase Margin

Phase margin as the change in open-loop phase shift required at unity gain to make a closed-loop system unstable.

From our previous example we know that this particular system will be unstable in closed loop if the Nyquist diagram encircles the -1 point. However, we must also realize that if the diagram is shifted by θ degrees, it will then touch the -1 point at the negative real axis, making the system marginally stable in closed loop. Therefore, the angle required to make this system marginally stable in closed loop is called the phase margin (measured in degrees). In order to find the point we measure this angle from, we draw a circle with radius of 1, find the point in the Nyquist diagram with a magnitude of 1 (gain of zero dB), and measure the phase shift needed for this point to be at an angle of 180 deg.



The Nyquist Stability Criterion

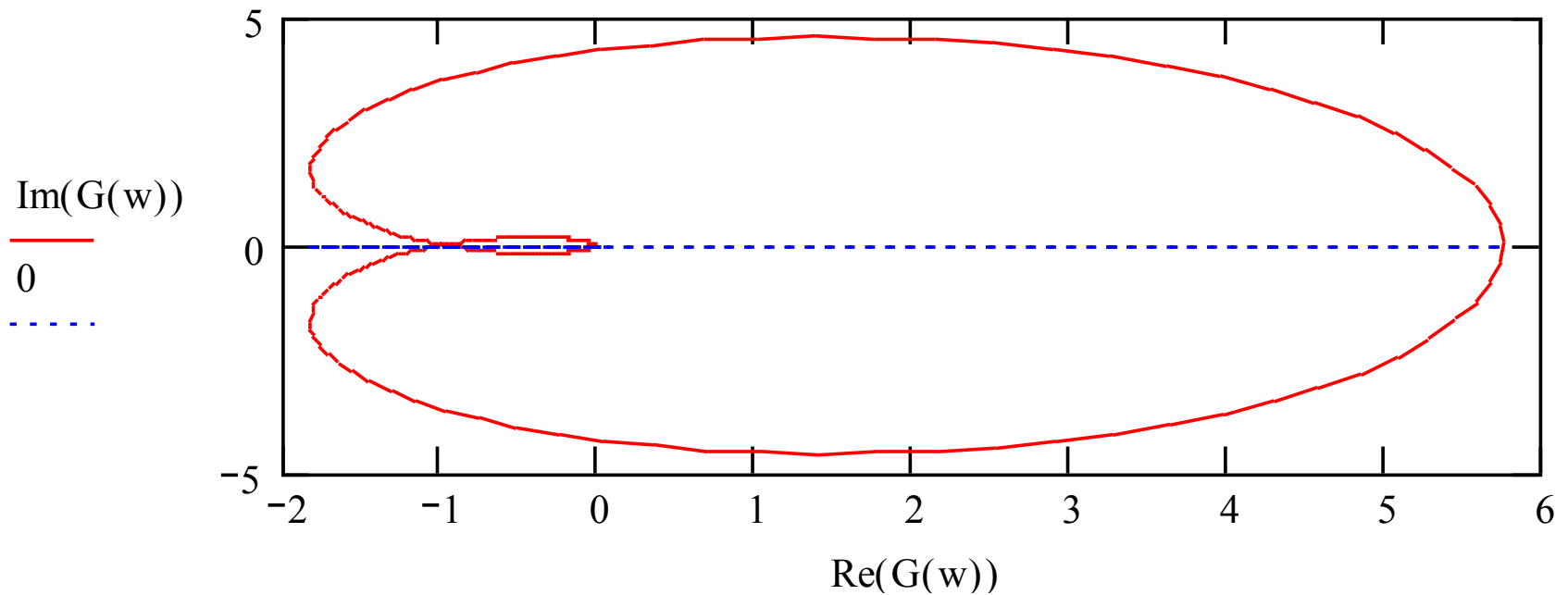
$$w := -100, -99.9..100$$

$$j := \sqrt{-1}$$

$$s(w) := j \cdot w$$

$$f(w) := -1$$

$$G(w) := \frac{50 \cdot 4.6}{s(w)^3 + 9 \cdot s(w)^2 + 30 \cdot s(w) + 40}$$



Consider the Negative Feedback System

Remember from the Cauchy criterion that the number N of times that the plot of $G(s)H(s)$ encircles -1 is equal to the number Z of zeros of $1 + G(s)H(s)$ enclosed by the frequency contour minus the number P of poles of $1 + G(s)H(s)$ enclosed by the frequency contour ($N = Z - P$).

Keeping careful track of open- and closed-loop transfer functions, as well as numerators and denominators, you should convince yourself that:

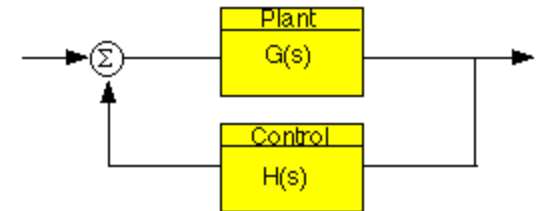
- the zeros of $1 + G(s)H(s)$ are the poles of the closed-loop transfer function
- the poles of $1 + G(s)H(s)$ are the poles of the open-loop transfer function.

The Nyquist criterion then states that:

- P = the number of open-loop (unstable) poles of $G(s)H(s)$
- N = the number of times the Nyquist diagram encircles -1
- clockwise encirclements of -1 count as positive encirclements
- counter-clockwise (or anti-clockwise) encirclements of -1 count as negative encirclements
- Z = the number of right half-plane (positive, real) poles of the closed-loop system

The important equation which relates these three quantities is:

$$\mathbf{Z = P + N}$$

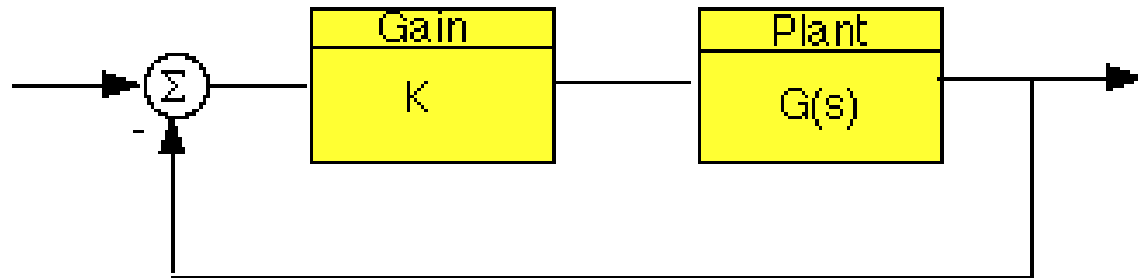


The Nyquist Stability Criterion - Application

Knowing the number of right-half plane (unstable) poles in open loop (P), and the number of encirclements of -1 made by the Nyquist diagram (N), we can determine the closed-loop stability of the system.

If $Z = P + N$ is a positive, nonzero number, the closed-loop system is unstable.

We can also use the Nyquist diagram to find the range of gains for a closed-loop unity feedback system to be stable. The system we will test looks like this:



where $G(s)$ is :

$$\frac{s^2 + 10s + 24}{s^2 - 8s + 15}$$

The Nyquist Stability Criterion

This system has a gain K which can be varied in order to modify the response of the closed-loop system. However, we will see that we can only vary this gain within certain limits, since we have to make sure that our closed-loop system will be stable. This is what we will be looking for: the range of gains that will make this system stable in the closed loop.

The first thing we need to do is find the number of positive real poles in our open-loop transfer function:

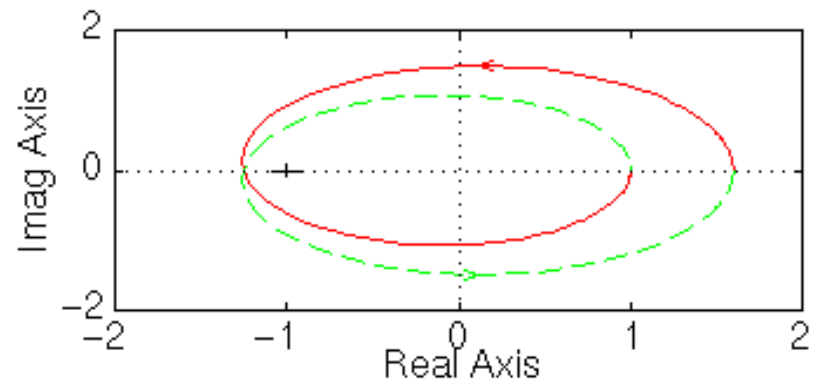
```
roots([1 -8 15])
ans =
    5
    3
```

The poles of the open-loop transfer function are both positive. Therefore, we need two anti-clockwise ($N = -2$) encirclements of the Nyquist diagram in order to have a stable closed-loop system ($Z = P + N$). If the number of encirclements is less than two or the encirclements are not anti-clockwise, our system will be unstable.

Let's look at our Nyquist diagram for a gain of 1:

```
nyquist([ 1 10 24], [ 1 -8 15])
```

There are two anti-clockwise encirclements of -1. Therefore, the system is stable for a gain of 1.



The Nyquist Stability Criterion

MathCAD Implementation

$$w := -100, -99.9..100$$

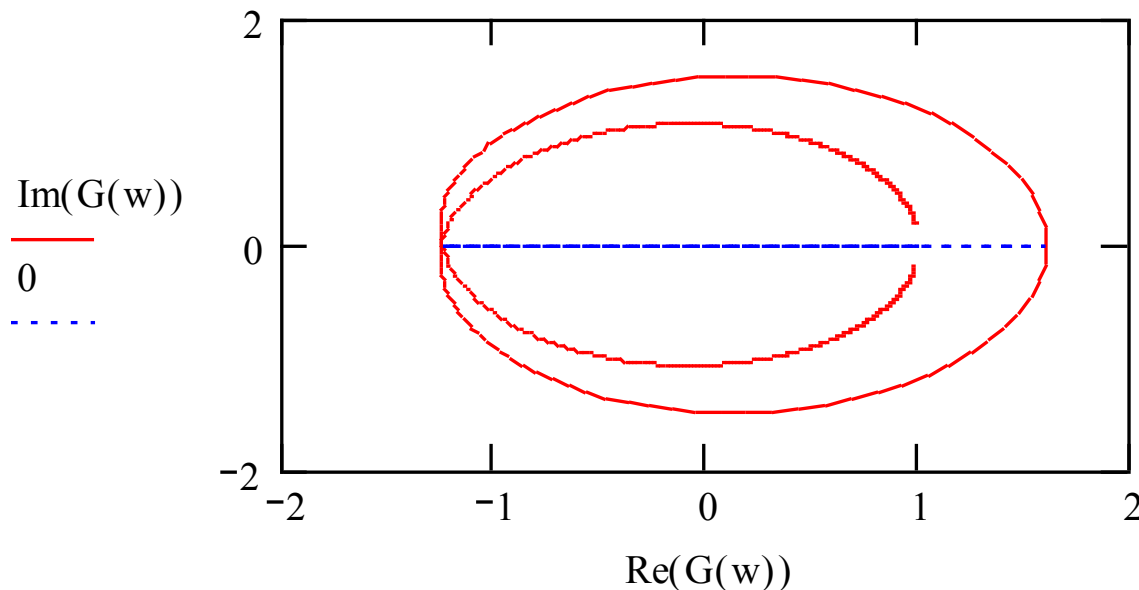
$$j := \sqrt{-1}$$

$$s(w) := j \cdot w$$

$$G(w) := \frac{s(w)^2 + 10 \cdot s(w) + 24}{s(w)^2 - 8 \cdot s(w) + 15}$$

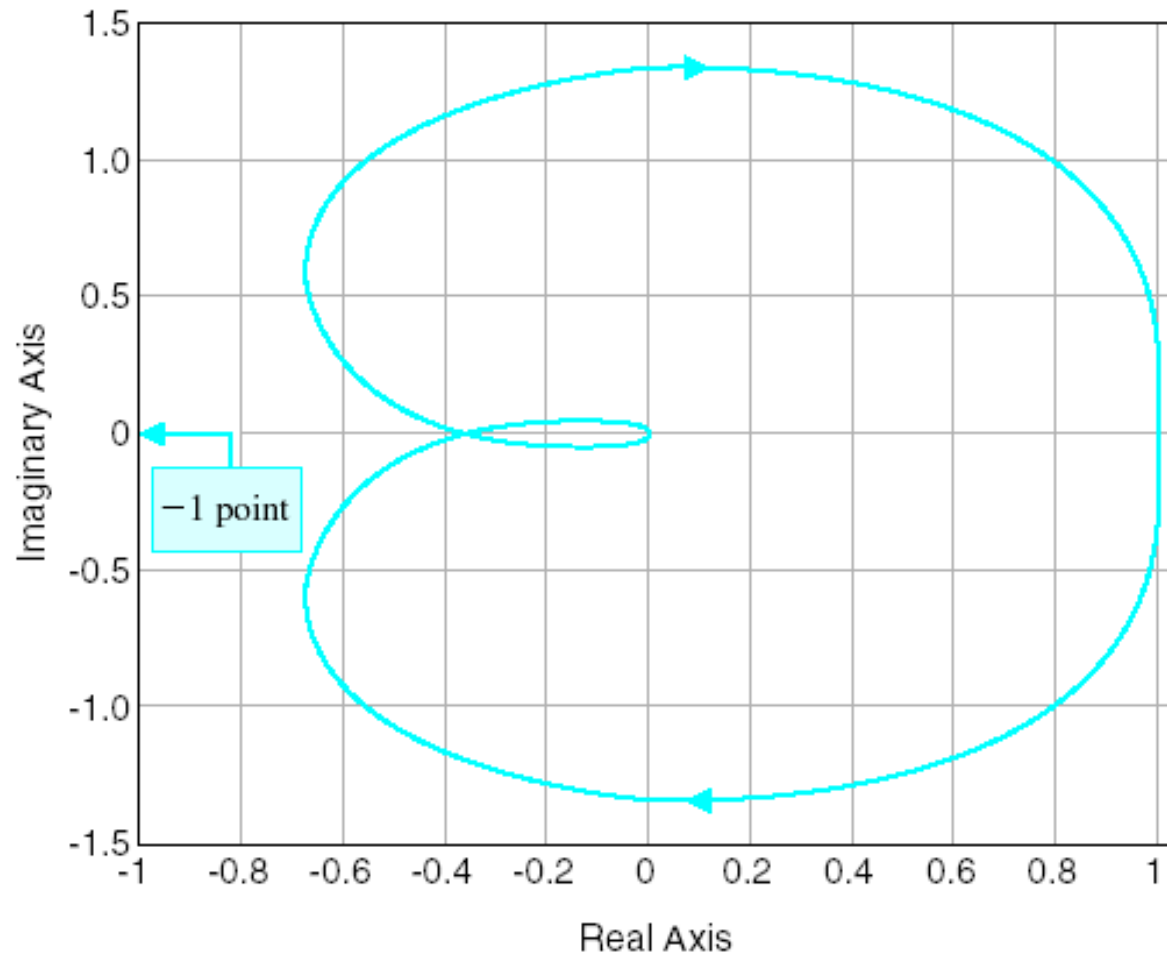
There are two anti-clockwise encirclements of -1.

Therefore, the system is stable for a gain of 1.



The Nyquist Stability Criterion

```
>>num=[0.5]; den=[1 2 1 0.5];  
>>sys=tf(num,den);  
>>nyquist(sys)
```



Time-Domain Performance Criteria Specified In The Frequency Domain

Open and closed-loop frequency responses are related by:

$$T(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)}$$

$$M_{pw} = \frac{1}{2 \cdot \zeta \cdot \sqrt{1 - \zeta^2}} \quad \zeta < 0.707$$

$$G(\omega) = u + j \cdot v \quad M = M(\omega)$$

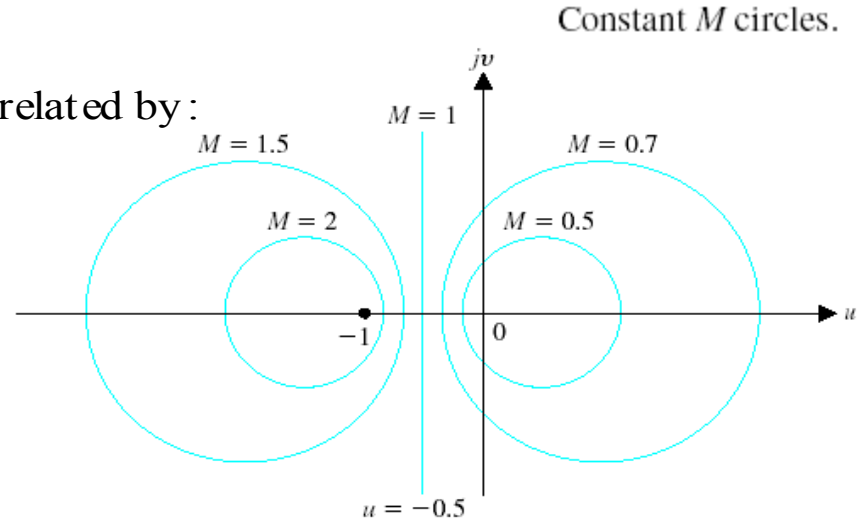
$$M(\omega) = \left| \frac{G(j\omega)}{1 + G(j\omega)} \right| = \left| \frac{u + jv}{1 + u + jv} \right| = \frac{\sqrt{u^2 + v^2}}{\sqrt{(1 + u)^2 + v^2}}$$

Squaring and rearranging

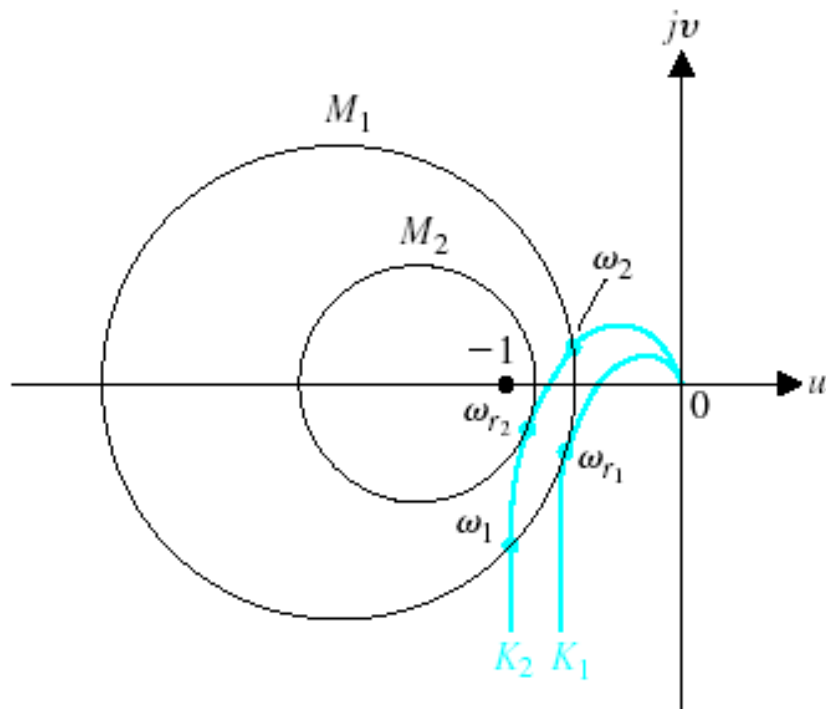
$$\left(u - \frac{M^2}{1 - M^2} \right)^2 + v^2 = \left(\frac{M}{1 - M^2} \right)^2$$

which is the equation of a circle on u - v plane with a center at

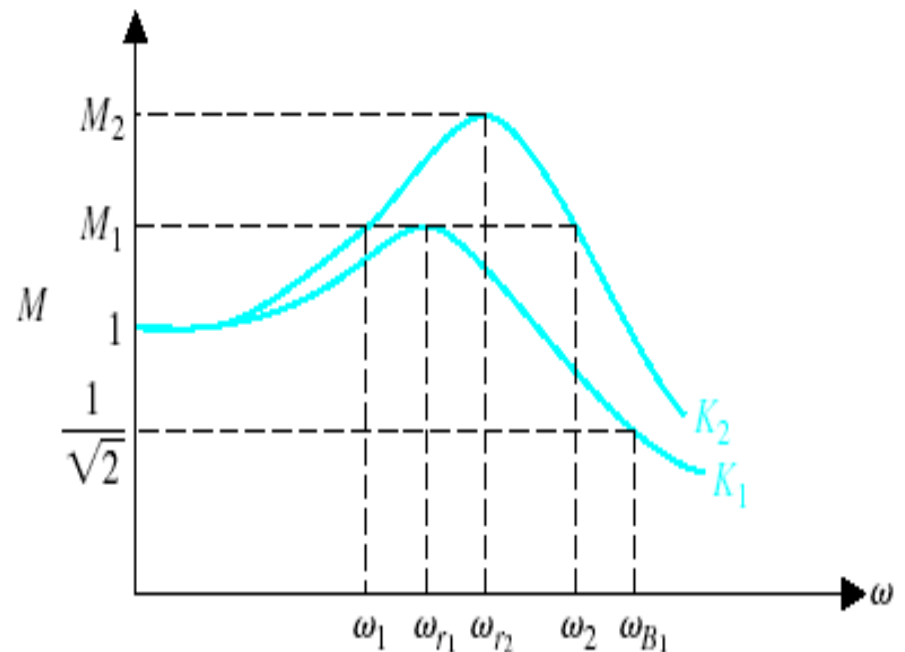
$$u = \frac{M^2}{1 - M^2} \quad v = 0$$



Time-Domain Performance Criteria Specified In The Frequency Domain



Polar plot of $G(j\omega)$ for two values of a gain ($K_2 > K_1$).



Closed-loop frequency response of $T(j\omega) = G(j\omega)/(1 + G(j\omega))$.
Note that $K_2 > K_1$.

The Nichols Stability Method

Polar Stability Plot - Nichols Mathcad Implementation

This example makes a polar plot of a transfer function and draws one contour of constant closed-loop magnitude. To draw the plot, enter a definition for the transfer function $G(s)$:

$$G(s) := \frac{45000}{s \cdot (s + 2) \cdot (s + 30)}$$

The frequency range defined by the next two equations provides a logarithmic frequency scale running from 1 to 100. You can change this range by editing the definitions of m and ω_m :

$$m := 0..100 \quad \omega_m := 10^{0.02 \cdot m}$$

Now enter a value for M to define the closed-loop magnitude contour that will be plotted.

$$M := 1.1$$

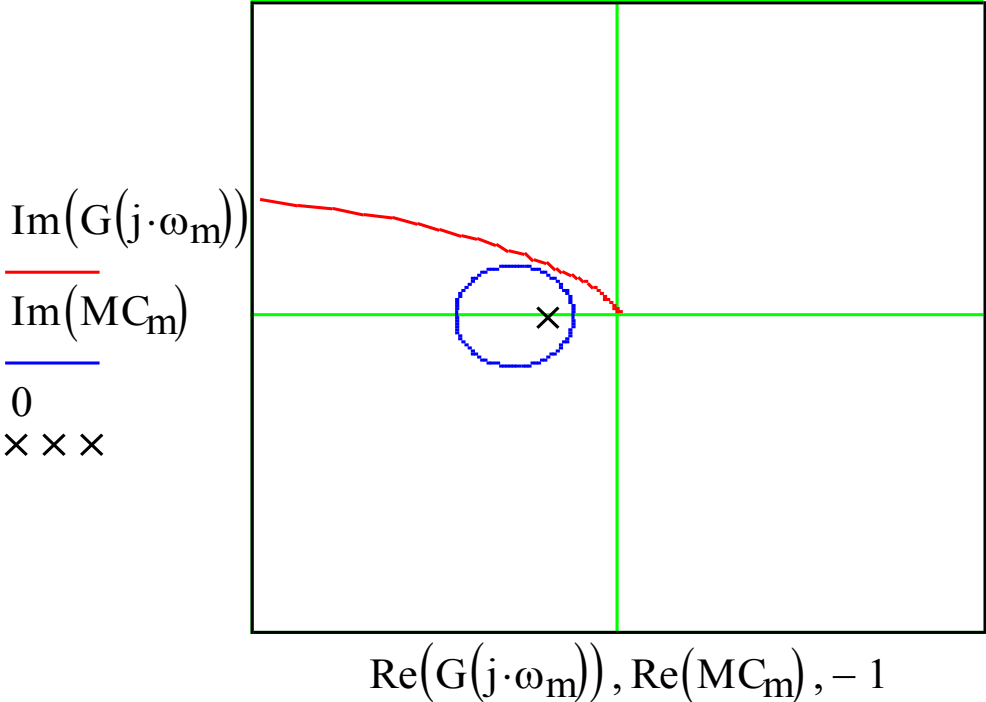
Calculate the points on the M-circle:

$$MC_m := \left(\frac{-M^2}{M^2 - 1} + \left| \frac{M}{M^2 - 1} \right| \cdot \exp(2 \cdot \pi \cdot j \cdot 0.01 \cdot m) \right)$$

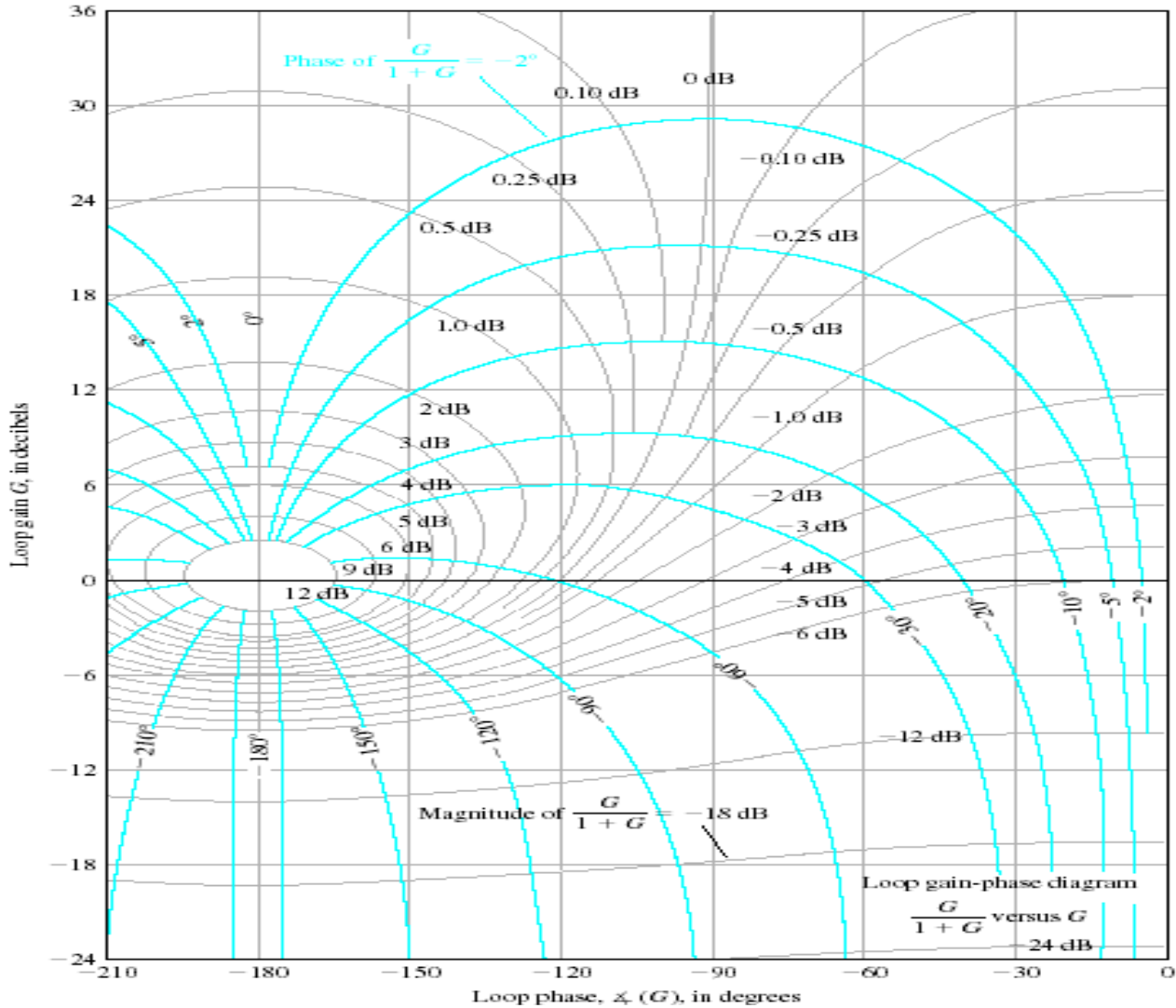
The first plot shows G , the contour of constant closed-loop magnitude M .

The Nichols Stability Method

The first plot shows G , the contour of constant closed-loop magnitude M , and the Nyquist of the open loop system



The Nichols Stability Method



Nichols chart. The phase curves for the closed-loop system are shown in color.

The Nichols Stability Method

$$G(\omega) := \frac{1}{j \cdot \omega \cdot (j \cdot \omega + 1) \cdot (0.2 \cdot j \cdot \omega + 1)}$$

$$M_{pw} := 2.5 \text{ dB} \quad \omega_r := 0.8$$

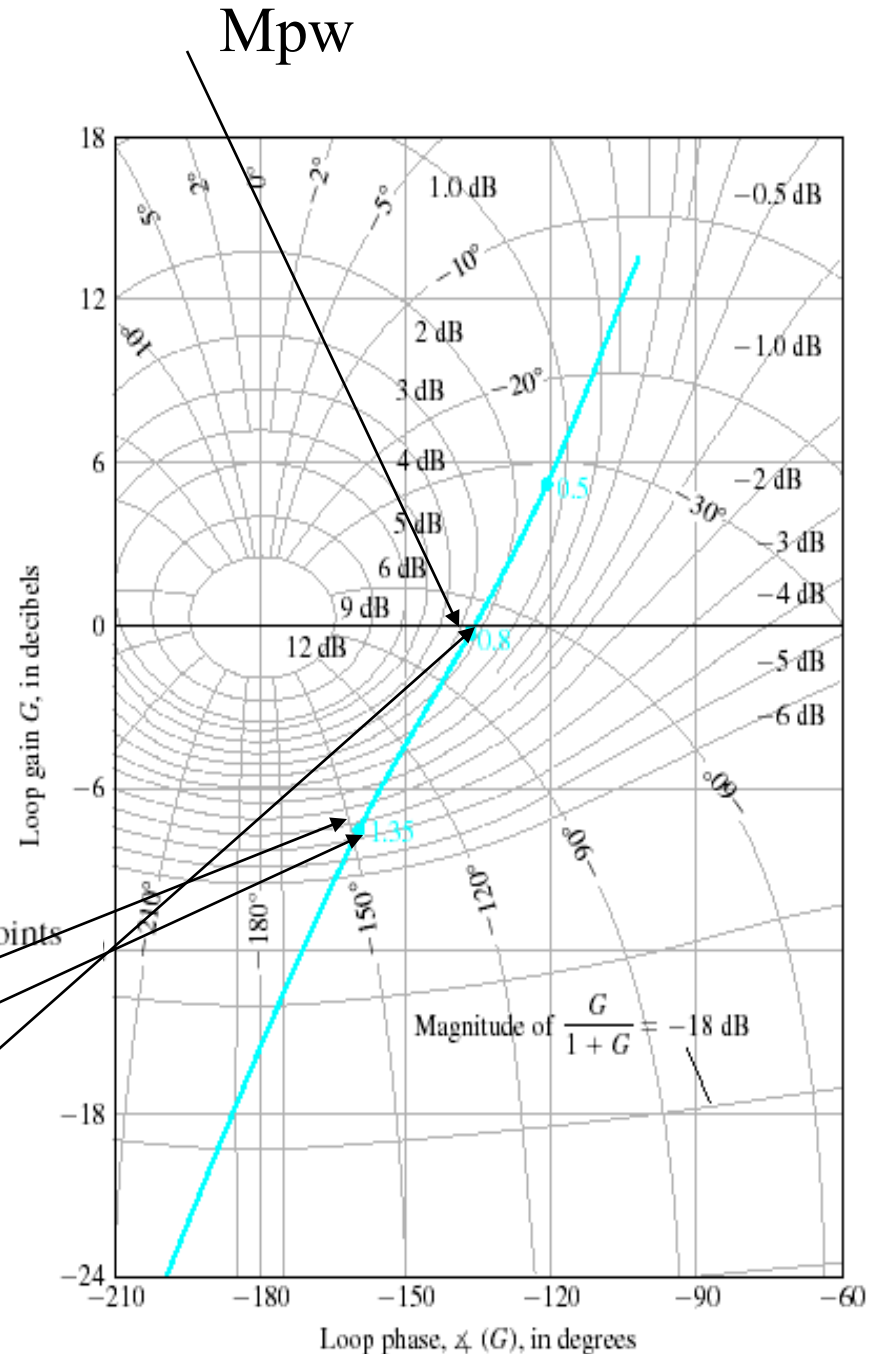
The closed-loop phase angle

at ω_r is equal to -72 degrees and $\omega_b = 1.33$

The closed-loop phase angle at ω_b is equal to -142 degrees

Nichols diagram for $G(j\omega) = 1/j\omega(j\omega + 1)(0.2j\omega + 1)$. Three points on curve are shown for $\omega = 0.5, 0.8$, and 1.35 , respectively.

-3dB
 -142 deg
 -72 deg
 $\omega_r = 0.8$



The Nichols Stability Method

$$G(\omega) := \frac{0.64}{j \cdot \omega \cdot [(j \cdot \omega)^2 + j \cdot \omega + 1]}$$

Phase Margin = 30 degrees

On the basis of the phase we estimate $\zeta := 0.30$

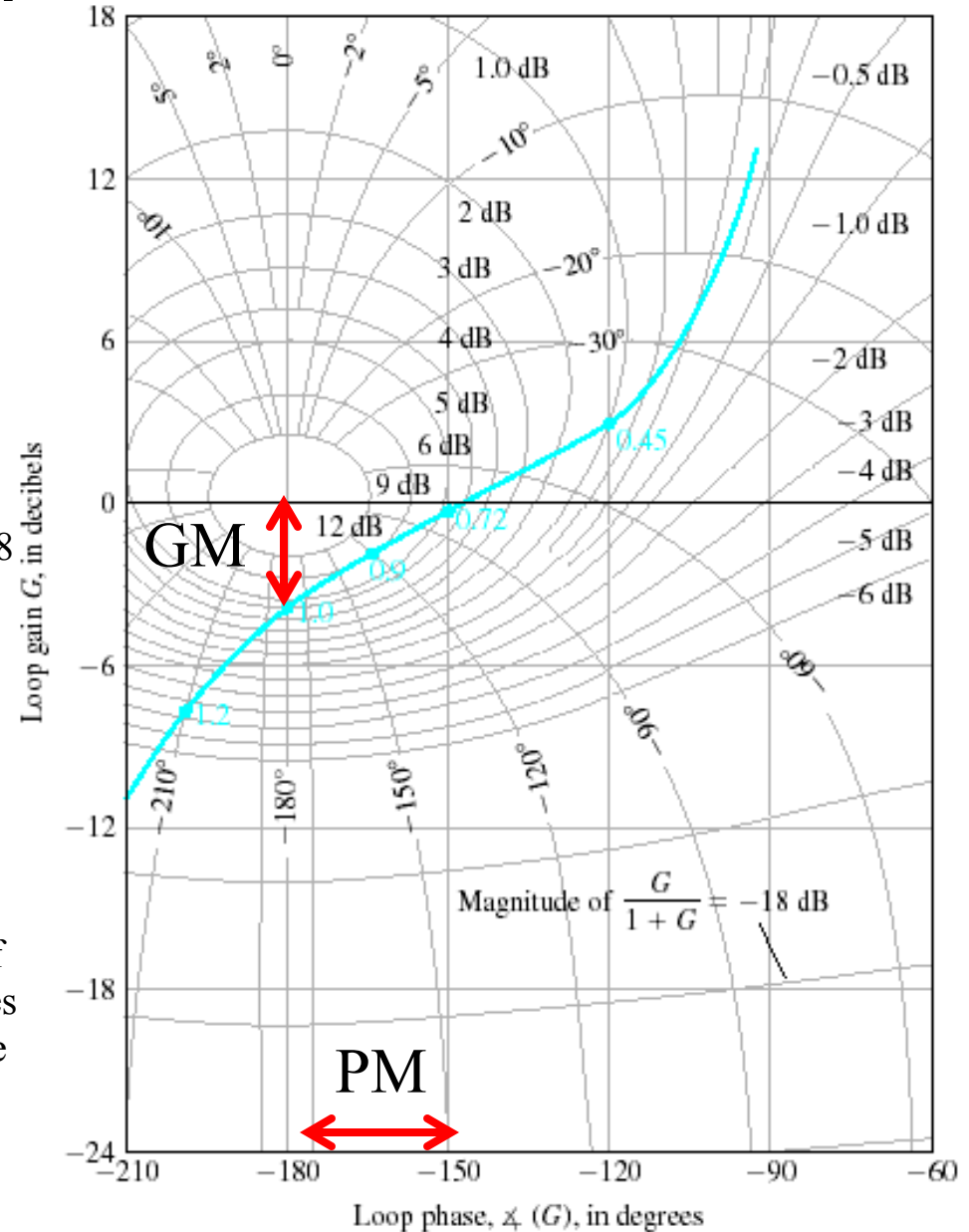
$$M_{pw} := 9 \text{ dB} \quad M_{pw} := 2.8 \quad \omega_r := 0.88$$

From equation

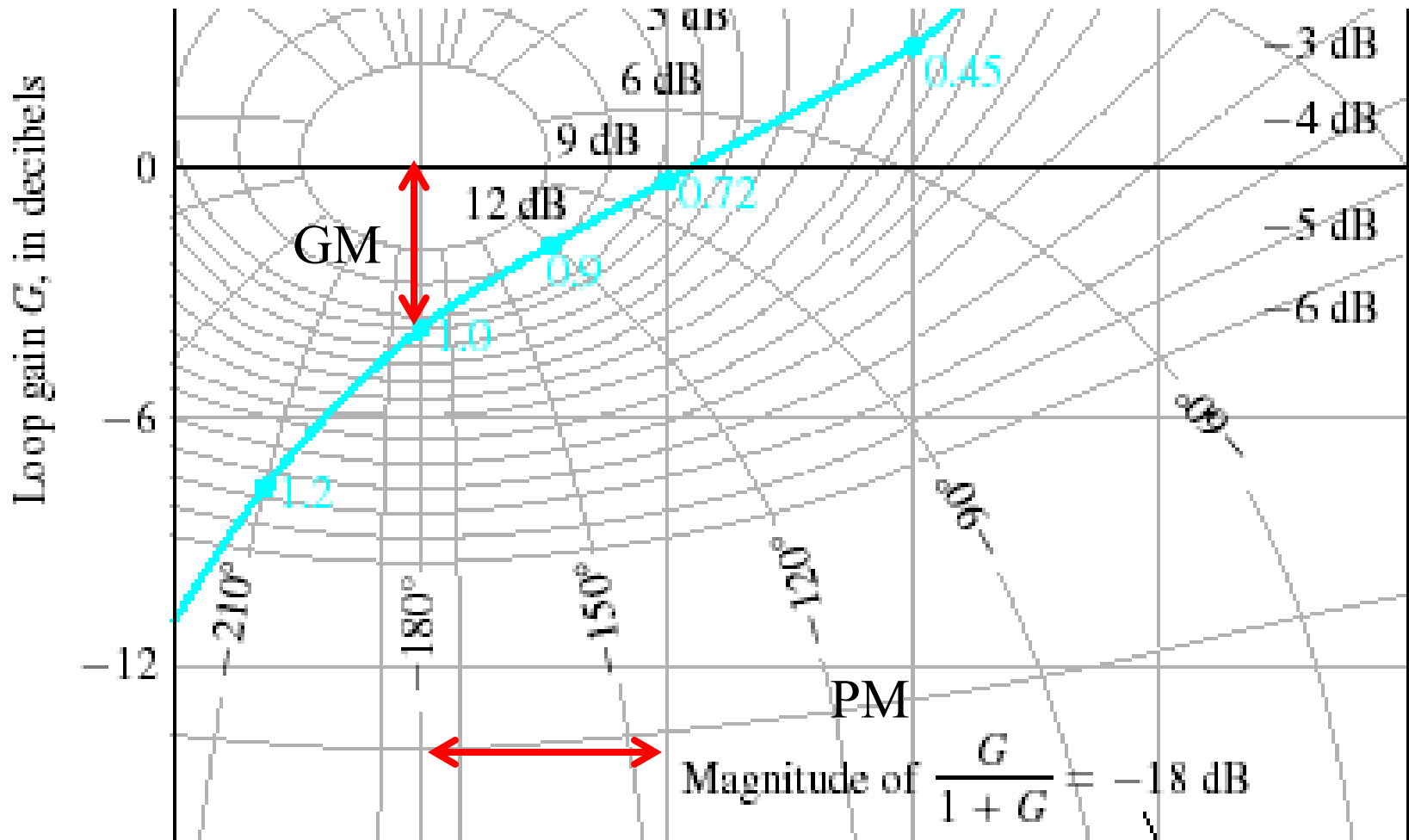
$$M_{pw} = \frac{1}{2 \cdot \zeta \cdot \sqrt{1 - \zeta^2}} \quad \zeta := 0.18$$

We are confronted with conflicting

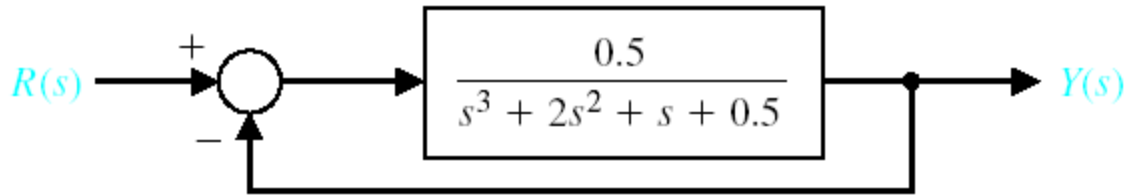
The apparent conflict is caused by the nature of $G(j\omega)$ which slopes rapidly toward 180 degrees line from the 0-dB axis. The designer must use the frequency-domain-time-domain correlation with caution



The Nichols Stability Method

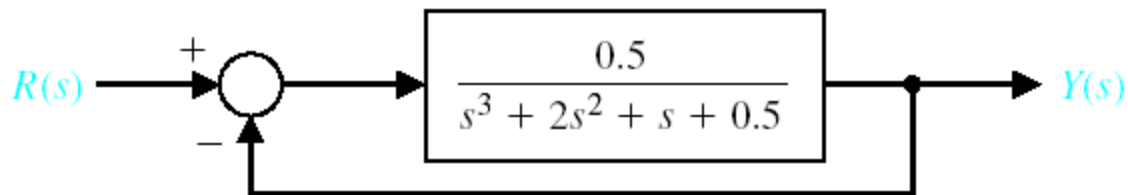


Examples – Bode and Nyquist



A closed-loop control system example for Nyquist and Bode with relative stability.

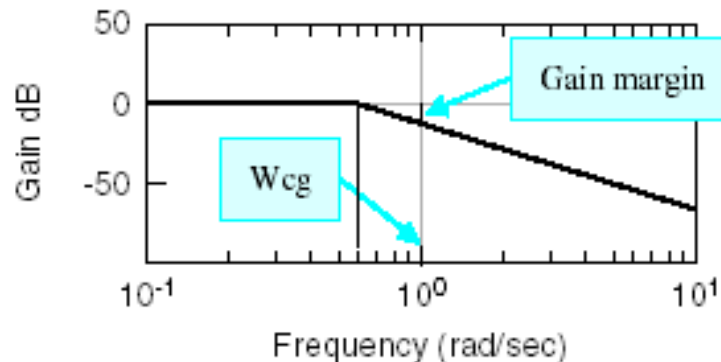
Examples - Bode



```
[mag,phase,w]=bode(sys);  
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
```

or

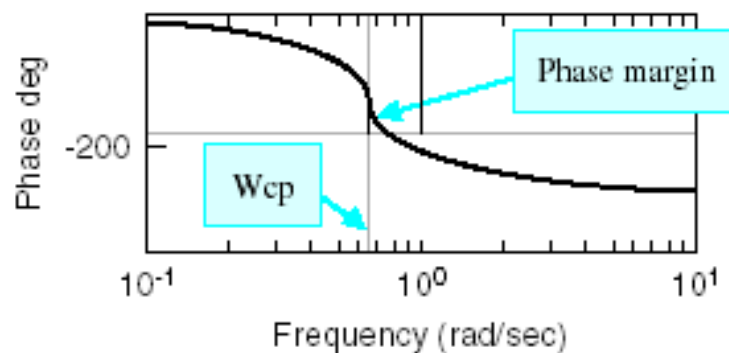
```
[Gm,Pm,Wcg,Wcp]=margin(sys);
```



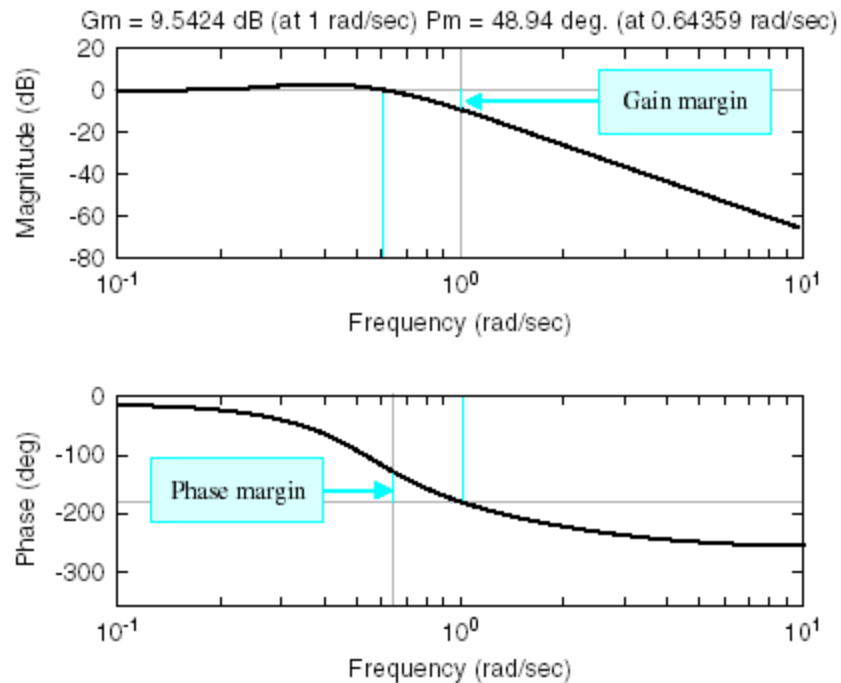
Example

```
num=[0.5]; den=[1 2 1 0.5];  
sys=tf(num,den);  
margin(sys);
```

Gm = gain margin (dB)
Pm = phase margin (deg)
Wcg = freq. for phase = -180
Wcp = freq. for gain = 0 dB



Examples - Bode

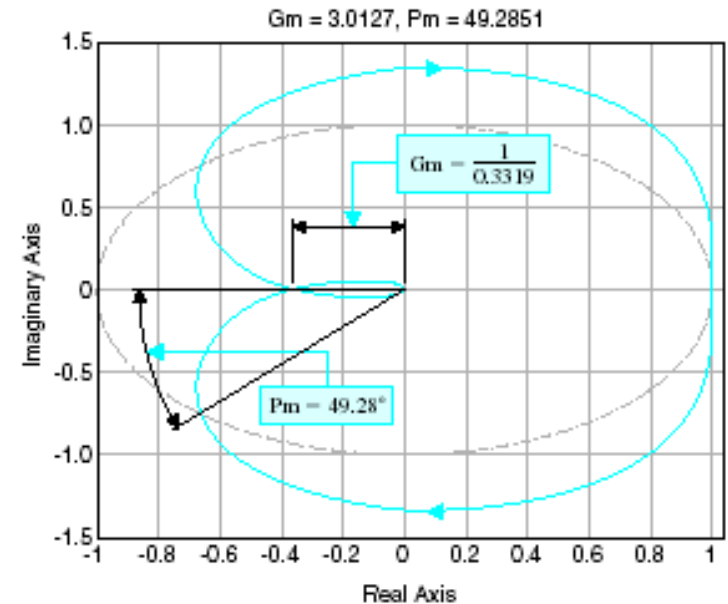
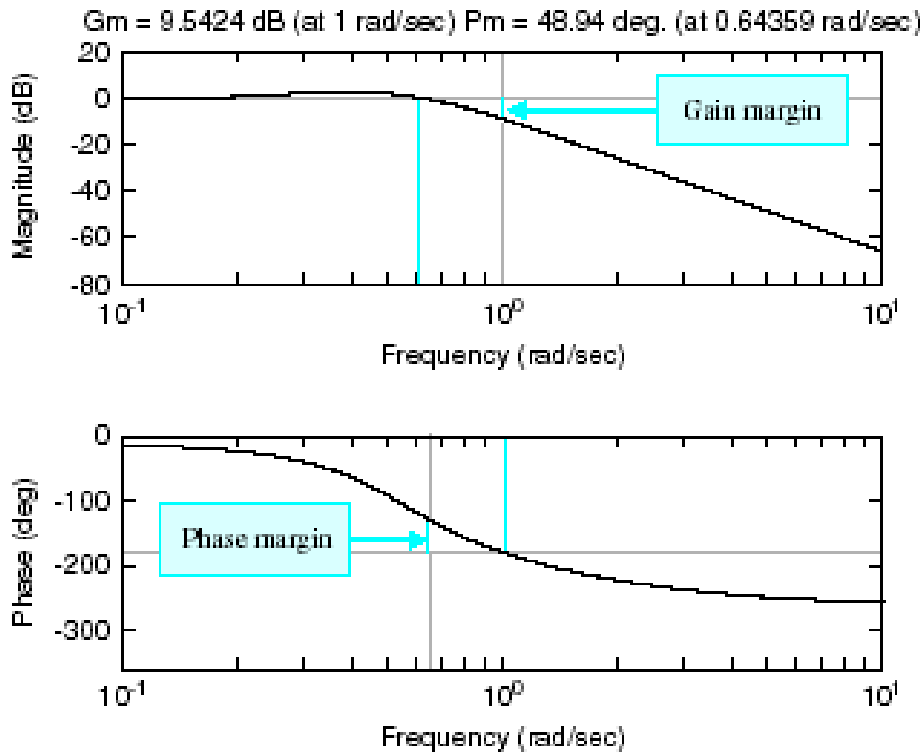


```
num=[0.5];  
den=[1 2 1 0.5];  
sys=tf(num,den);  
%  
w=logspace(-1,1,200);  
%  
[mag,phase,w]=bode(sys,w);  
%  
margin(mag,phase,w);
```

Open-loop system

Specify frequency range

Examples – Bode and Nyquist



```

num=[0.5];
den=[1 2 1 0.5];
sys=tf(num,den);
%
w=logspace(-1,1,200);
%
[mag,phase,w]=bode(sys,w);
%
margin(mag,phase,w);
    
```

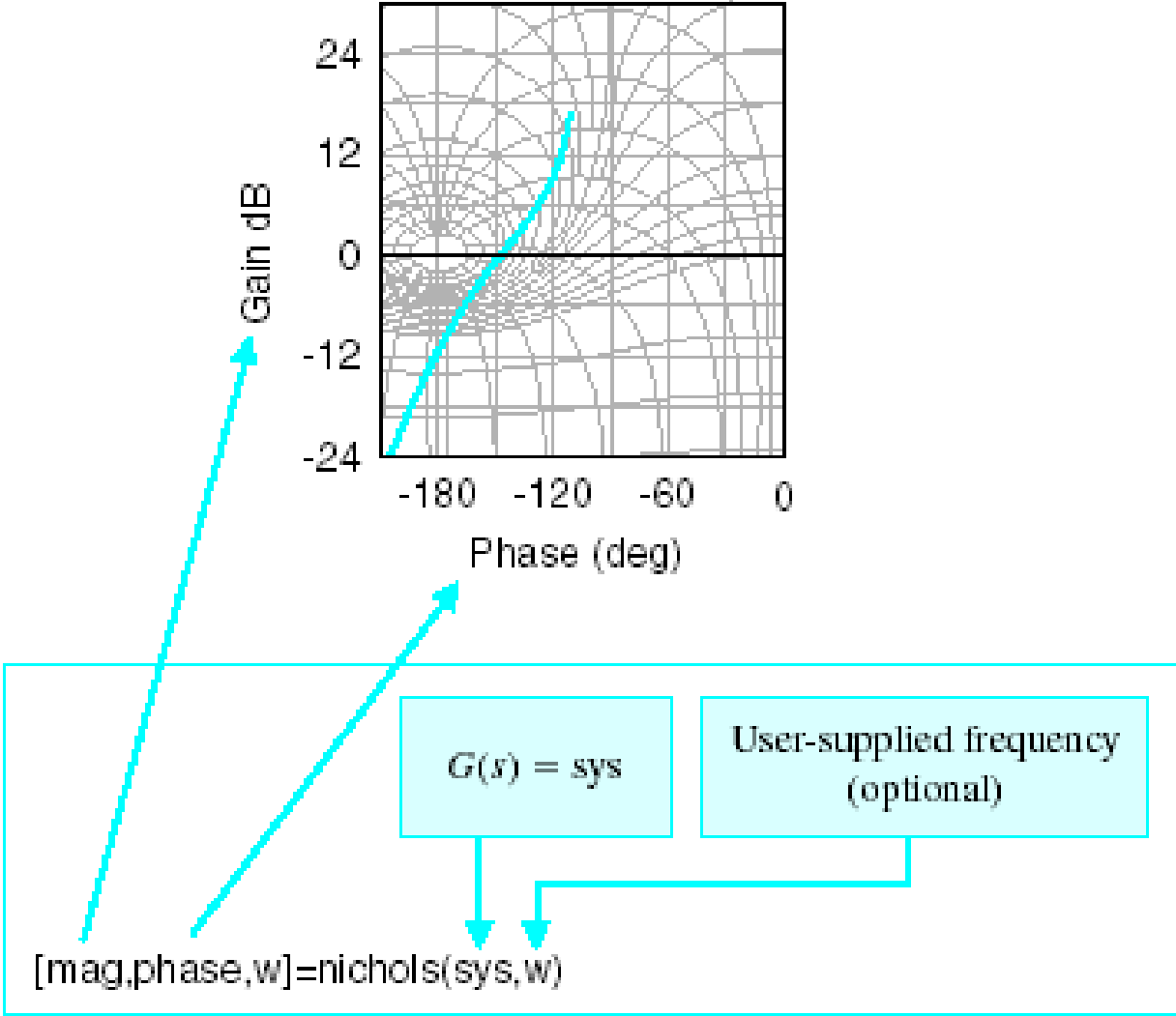
Open-loop system

Specify frequency range

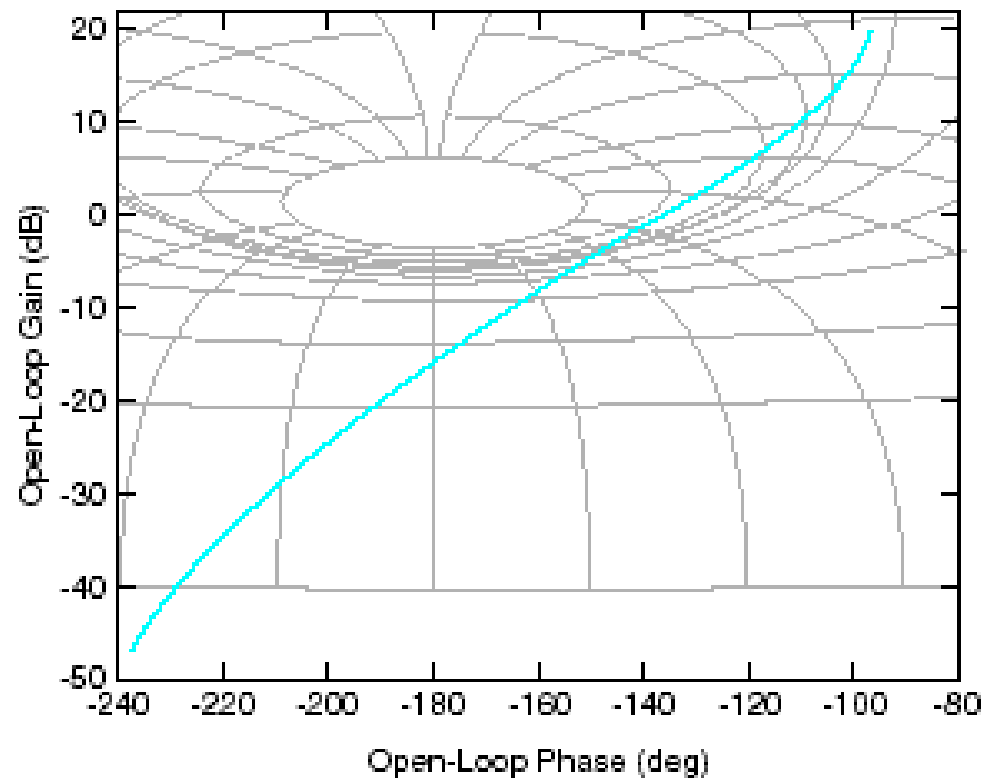
```

% The Nyquist plot of
%
%           0.5
% G(s) = -----
%          s^3 + 2 s^2 + s + 0.5
%
% Compute gain and phase margins.
% with gain and phase margin calculation.
num=[0.5]; den=[1 2 1 0.5]; sys=tf(num,den);
%
[mag,phase,w]=bode(sys);
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
%
nyquist(sys);
title(['Gm = ',num2str(gm),' Pm = ',num2str(Pm)])
%
Nyquist plot
Label gain and phase margins on plot.
    
```

Examples - Nichols



Examples - Nichols



```
num=[1]; den=[0.2 1.2 1 0];  
sys=tf(num,den);  
w=logspace(-1,1,400);  
nichols(sys,w);  
ngrid
```

Set up to generate
Fig. 9.27

Plot Nichols chart
and add grid lines.