Conversion of state variable model to transfer function model and vice-versa

State Space Representation (noise free linear systems)

- **State Space form**

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

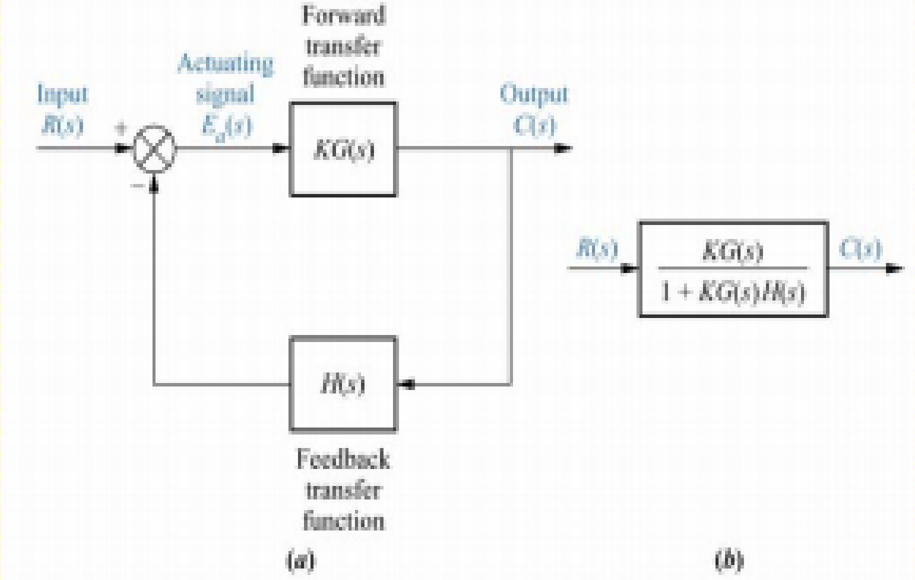
A - System matrix- *n x n*

B - Input matrix- *n x m*

C - Output matrix- *p x n*

D - Feed forward matrix – *p x m*

- **Transfer Function form**



Q: Is conversion between the two forms possible?

A: Yes.

State Space to Transfer Function

- Known:

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

- Taking Laplace transform (with zero initial conditions)

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

- The state equation can be placed in the form

$$(sI - A)X(s) = BU(s)$$

- Pre-multiplying both sides by $(sI - A)^{-1}$

$$X(s) = (sI - A)^{-1}BU(s)$$

- Substituting for $X(s)$ in the output equation,

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{\text{Transfer Function Matrix } T(s)}U(s)$$

Example

- State space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

- Using the expression for derived transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

$$T(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{s(s+3)+2} \right) \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = \frac{1}{s^2+3s+2}$$

Transfer Function to State Space

- The process of converting transfer function to state space form is **<u>NOT</u>** unique. There are various "realizations" possible.
- All realizations are "equivalent" (i.e. properties do not change). However, one representation may have some advantages over others for a particular task.
- Possible representations:
  - First companion form (controllable canonical form)
  - Jordan canonical form
  - Alternate first companion form (Toeplitz first companion form)
  - Second companion form (observable canonical form)

First Companion Form: SISO Case
(Controllable canonical form)

$$H(s) = \frac{y(s)}{u(s)} = \left[\frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \right]$$

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = u$$

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \quad \cdots \quad + a_n y = u$$

First Companion Form: SISO Case
(Controllable canonical form)

Choose output $y(t)$ and its $(n-1)$ derivatives as

$$\begin{bmatrix} x_1 = y \\ x_2 = \dfrac{dy}{dt} \\ \vdots \\ x_n = \dfrac{d^{n-1}y}{dt^{n-1}} \end{bmatrix} \xrightarrow{\text{differentiating}} \begin{bmatrix} \dot{x}_1 = \dfrac{dy}{dt} \\ \dot{x}_2 = \dfrac{d^2y}{dt^2} \\ \vdots \\ \dot{x}_n = \dfrac{d^n y}{dt^n} \end{bmatrix}$$

First Companion Form: SISO Case
(Controllable canonical form)

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & \cdots & \cdots & \cdots & -a_1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} +
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [0]u
$$