FEMT Unit1:

Vector Fields

Scalar

- **Scalar**: A quantity that has only magnitude.
- For example time, mass, distance, temperature and population are scalars.
- Scalar is represented by a letter – e.g., A, B

Vector

- **Vector**: A quantity that has both magnitude and direction.
- Example: Velocity, force, displacement and electric field intensity.
- Vector is represent by a letter such as **A, B,** $\vec{A}$ or $\vec{\psi}$
- It can also be written as $\vec{A} = A\hat{a}$

where A is $\left|\vec{A}\right|$ which is the magnitude and $\hat{a}$ is unit vector

Unit Vector

- A unit vector along **A** is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along **A**.

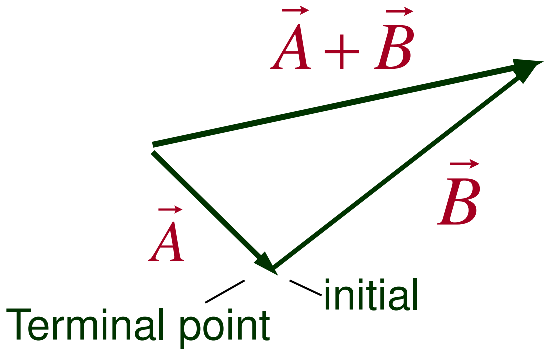
- It can be written as $\mathbf{a}_A$ or $\hat{a}$

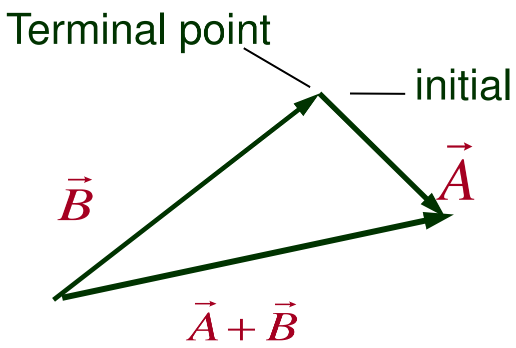
$$a_A = \frac{A}{|\vec{A}|} = \frac{A}{\vec{A}}$$

Thus $A = \vec{A} a_A$

Vector Addition

- The sum of two vectors for example vectors A and B can be obtain by moving one of them so that its terminal point (tip) coincides with the initial point (tail) of the other





Vector Subtraction

- Vector subtraction is similarly carried out as

  D = **A** − **B** = **A** + (-**B**)

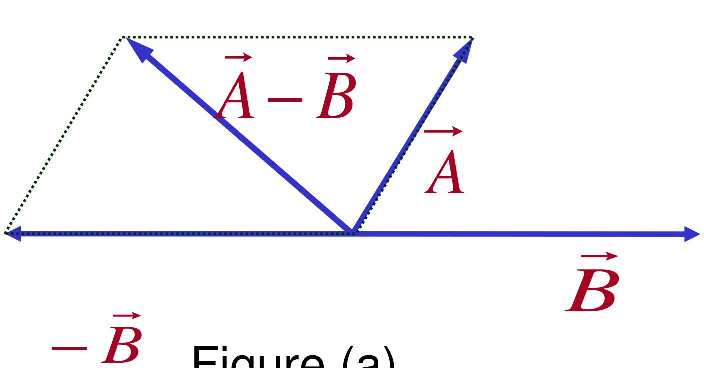


Figure (a)

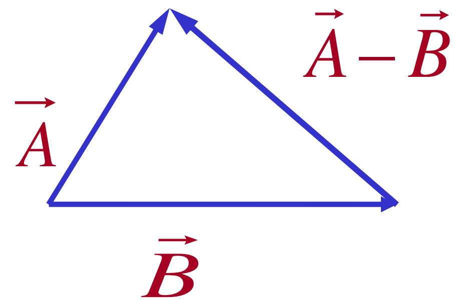


Figure (c)

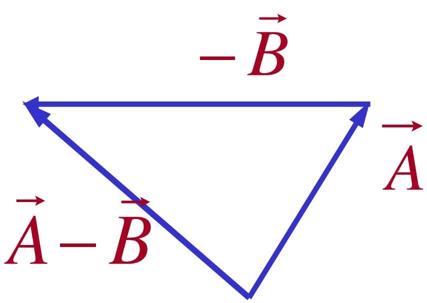


Figure (b)

Figure (c) shows that vector **D** is a vector that is must be added to **B** to give vector **A**

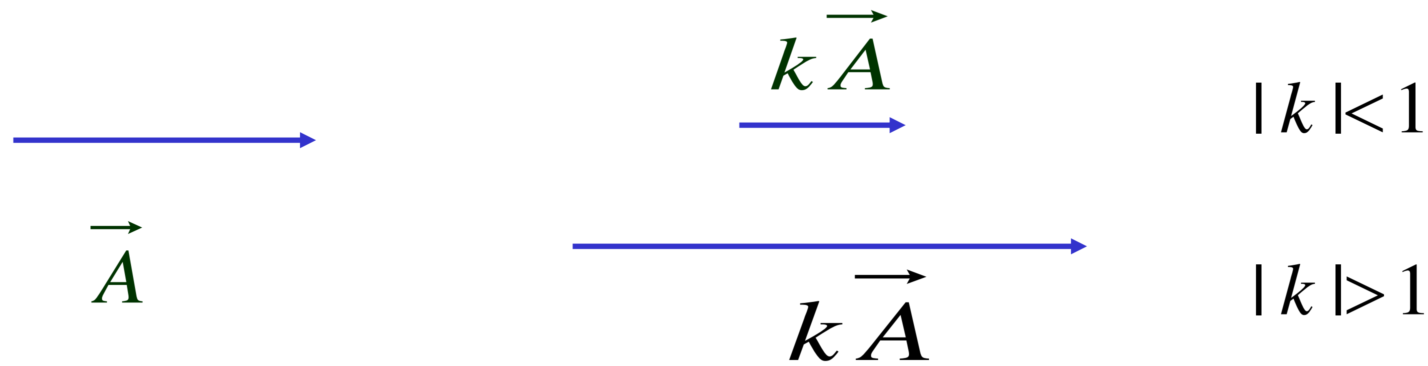
So if vector **A** and **B** are placed tail to tail then vector **D** is a vector that runs from the tip of **B** to **A**.

Vector multiplication

- Scalar (dot) product (**A●B**)
- Vector (cross) product (**A X B**)
- Scalar triple product **A ● (B X C)**
- Vector triple product **A X (B X C)**

Multiplication of a vector by a scalar

- Multiplication of a scalar *k* to a vector **A** gives a vector that points in the same direction as **A** and magnitude equal to |*kA*|



- The division of a vector by a scalar quantity is a multiplication of the vector by the reciprocal of the scalar quantity.

Scalar Product

- The dot product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} \bullet \vec{B}$ is defined as the product of the magnitude of $\vec{A}$ and $\vec{B}$, and the projection of $\vec{A}$ onto $\vec{B}$ (or vice versa).
- Thus ;

$$\overrightarrow{A} \bullet \overrightarrow{B} = |A||B| \cos\theta$$

Where θ is the angle between $\overrightarrow{A}$ and $\overrightarrow{B}$. The result of dot product is a **scalar quantity**.

Vector Product

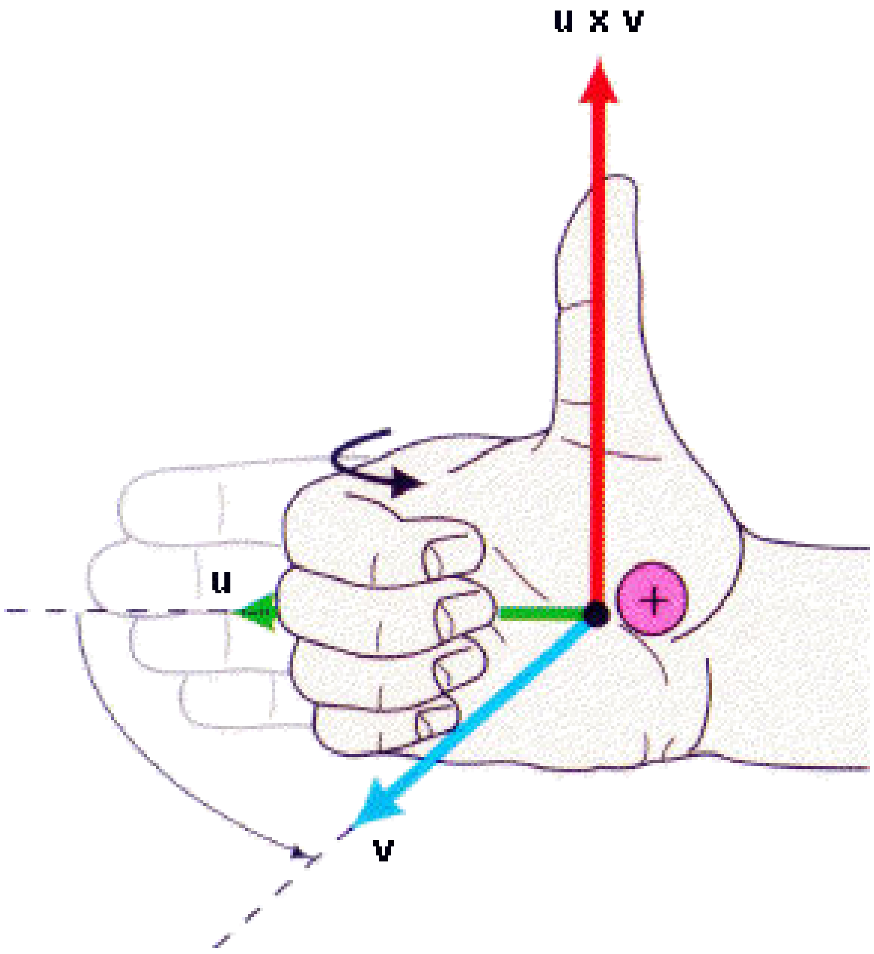
- The cross (or vector) product of two vectors A and B, written as is defined as

$$\vec{A} \times \vec{B} = |A||B| \sin \theta_{AB} \hat{n}$$

where; $\hat{n}$ a unit vector perpendicular to the plane that contains the two vectors. The direction of $\hat{n}$ is taken as the direction of the right thumb (using right-hand rule)

- The product of cross product is a **vector**

Right-hand Rule



Components of a vector

- A direct application of vector product is in determining the projection (or component) of a vector in a given direction. The projection can be scalar or vector.

- Given a vector **A**, we define the *scalar component* AB of **A** along vector **B** as

$$A_B = \boldsymbol{A} \cos \theta_{AB} = |A||a_B| \cos \theta_{AB}$$

or $A_B = \boldsymbol{A} \cdot \boldsymbol{a}_B$

Dot product

If $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$ then

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

which is obtained by multiplying A and B component by component.

- It follows that modulus of a vector is

$$|\vec{A}| = \sqrt{\vec{A} \bullet \vec{A}} = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$$

Cross Product

- If **A**=(A_x, A_y, A_z), **B**=(B_x, B_y, B_z) then

$$\vec{A}\times\vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} a_x + \begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} a_y + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} a_z$$

$$= (A_y B_z - A_z B_y) a_x + (A_z B_x - A_x B_z) a_y + (A_x B_y - A_y B_x) a_z$$

Cross Product

- Cross product of the unit vectors yield:

$$\mathbf{a}_{\mathbf{x}} \times \mathbf{a}_{\mathbf{y}} = \mathbf{a}_{\mathbf{z}}$$

$$\mathbf{a}_{\mathbf{y}} \times \mathbf{a}_{\mathbf{z}} = \mathbf{a}_{\mathbf{x}}$$

$$\mathbf{a}_{\mathbf{z}} \times \mathbf{a}_{\mathbf{a}} = \mathbf{a}_{\mathbf{y}}$$

Example 1

- Given three vectors **P** = $2a_x - a_z$

  Q = $2a_x - a_y + 2a_z$

  R = $2a_x - 3a_y + a_z$

Determine

a) (P+Q) X (P-Q)

b) Q•(R X P)

c) P•(Q X R)

d) $\sin\theta_{QR}$

e) P X(Q X R)

f) A unit vector perpendicular to both **Q** and **R**

Solution

(a)

$$\begin{aligned}
(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q}) &= \mathbf{P} \times (\mathbf{P} - \mathbf{Q}) + \mathbf{Q} \times (\mathbf{P} - \mathbf{Q}) \\
&= \mathbf{P} \times \mathbf{P} - \mathbf{P} \times \mathbf{Q} + \mathbf{Q} \times \mathbf{P} - \mathbf{Q} \times \mathbf{Q} \\
&= 0 + \mathbf{Q} \times \mathbf{P} + \mathbf{Q} \times \mathbf{P} - 0 \\
&= 2\mathbf{Q} \times \mathbf{P} \\
&= 2 \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} \\
&= 2(1 - 0)\, \mathbf{a}_x + 2(4 + 2)\, \mathbf{a}_y + 2(0 + 2)\, \mathbf{a}_z \\
&= 2\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z
\end{aligned}$$

Solution (cont')

(b) The only way $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$ makes sense is

$$\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = (2, -1, 2) \cdot \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (2, -1, 2) \cdot (3, 4, 6)$$

$$= 6 - 4 + 12 = 14.$$

Alternatively:

$$\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

Solution (cont')

To find the determinant of a 3 X 3 matrix, we repeat the first two rows and cross multiply; when the cross multiplication is from right to left, the result should be negated as shown below. This technique of finding a determinant applies only to a 3 X 3 matrix. Hence

$$\mathbf{Q}\cdot(\mathbf{R}\times\mathbf{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= +6 + 0 - 2 + 12 - 0 - 2$$

$$= 14$$

Solution (cont')

(c) From eq. (1.28)

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = 14$$

or

$$\begin{aligned}\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) &= (2, 0, -1) \cdot (5, 2, -4) \\ &= 10 + 0 + 4 \\ &= 14\end{aligned}$$

(d)

$$\sin \theta_{QR} = \frac{|\mathbf{Q} \times \mathbf{R}|}{|\mathbf{Q}||\mathbf{R}|} = \frac{|(5, 2, -4)|}{|(2, -1, 2)||(2, -3, 1)|}$$

$$= \frac{\sqrt{45}}{3\sqrt{14}} = \frac{\sqrt{5}}{\sqrt{14}} = 0.5976$$

Solution (cont')

(e)

$$\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = (2, 0, -1) \times (5, 2, -4)$$
$$= (2, 3, 4)$$

Alternatively, using the bac-cab rule,

$$\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = \mathbf{Q}(\mathbf{P} \cdot \mathbf{R}) - \mathbf{R}(\mathbf{P} \cdot \mathbf{Q})$$
$$= (2, -1, 2)(4 + 0 - 1) - (2, -3, 1)(4 + 0 - 2)$$
$$= (2, 3, 4)$$

(f) A unit vector perpendicular to both **Q** and **R** is given by

$$\mathbf{a} = \frac{\pm \mathbf{Q} \times \mathbf{R}}{|\mathbf{Q} \times \mathbf{R}|} = \frac{\pm (5, 2, -4)}{\sqrt{45}}$$
$$= \pm (0.745, 0.298, -0.596)$$

Note that $|\mathbf{a}| = 1$, $\mathbf{a} \cdot \mathbf{Q} = 0 = \mathbf{a} \cdot \mathbf{R}$. Any of these can be used to check **a**.

Solution (cont')

(g) The component of **P** along **Q** is

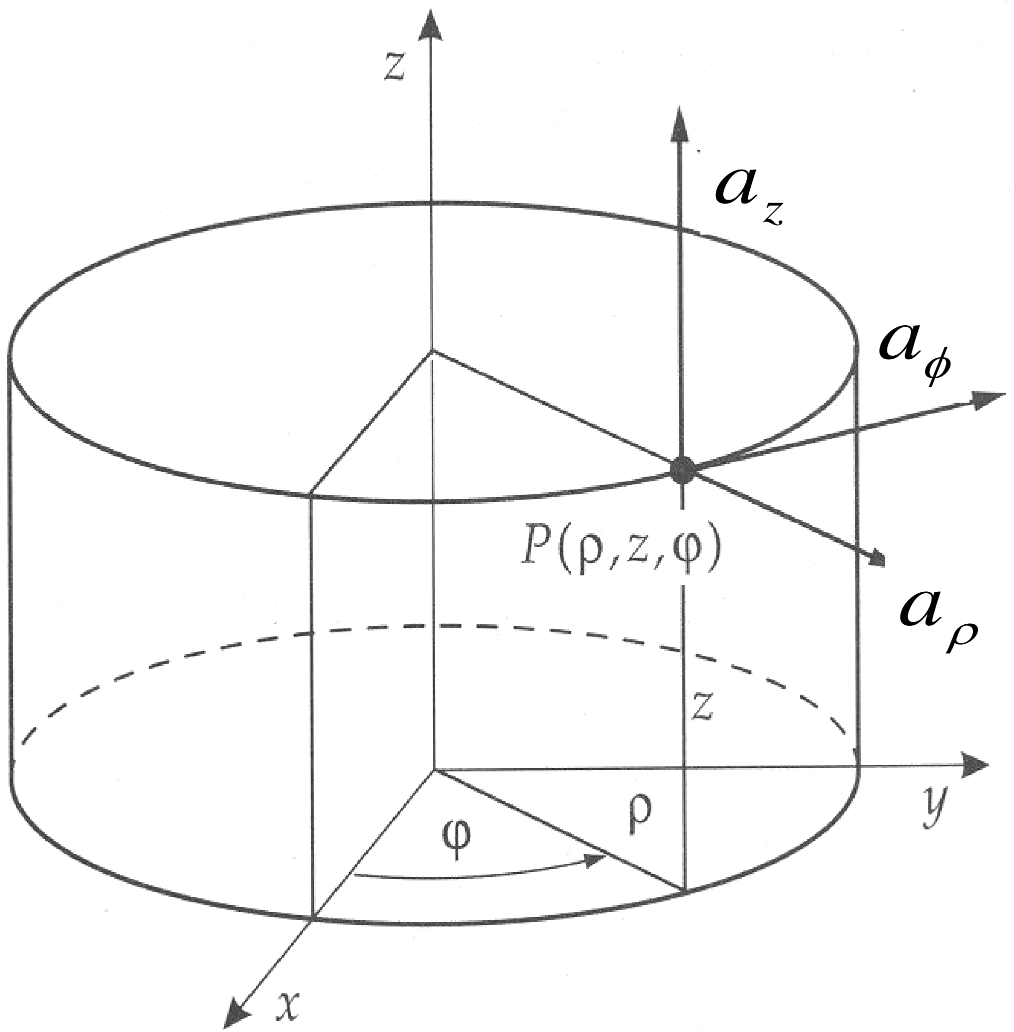
$$
\begin{aligned}
\mathbf{P}_Q &= |\mathbf{P}| \cos \theta_{PQ} \mathbf{a}_Q \\
&= (\mathbf{P} \cdot \mathbf{a}_Q)\mathbf{a}_Q = \frac{(\mathbf{P} \cdot \mathbf{Q})\mathbf{Q}}{|\mathbf{Q}|^2} \\
&= \frac{(4 + 0 - 2)(2, -1, 2)}{(4 + 1 + 4)} = \frac{2}{9}(2, -1, 2) \\
&= 0.4444\mathbf{a}_x - 0.2222\mathbf{a}_y + 0.4444\mathbf{a}_z.
\end{aligned}
$$

Cylindrical Coordinates

- Very convenient when dealing with problems having cylindrical symmetry.

- A point P in cylindrical coordinates is represented as (ρ, Φ, z) where
  - ρ: is the radius of the cylinder; radial displacement from the z-axis
  - Φ: *azimuthal* angle or the angular displacement from x-axis
  - z : vertical displacement z from the origin (as in the cartesian system).

Cylindrical Coordinates



Cylindrical Coordinates

- The range of the variables are

  $0 \le \rho < \infty$, $0 \le \Phi < 2\pi$, $-\infty < z < \infty$

- vector $\vec{A}$ in cylindrical coordinates can be written as (A_ρ, A_ϕ, A_z) or $A_\rho a_\rho + A_\phi a_\phi + A_z a_z$

- The magnitude of $\vec{A}$ is

$$|\vec{A}| = \sqrt{A_\rho{}^2 + A_\phi{}^2 + A_z{}^2}$$

Relationships Between Variables

- The relationships between the variables (x,y,z) of the Cartesian coordinate system and the cylindrical system (ρ, φ , z) are obtained as

$$\rho = \sqrt{x^2 + y^2} \qquad x = \rho\cos\phi$$

$$\phi = \tan^{-1} y/x \qquad y = \rho\sin\phi$$

$$z = z \qquad z = z$$

- So a point P (3, 4, 5) in Cartesian coordinate is the same as?

Relationships Between Variables

$$\rho = \sqrt{3^2 + 4^2} = 5$$

$$\phi = \tan^{-1} 4/3 = 0.927\, rad$$

$$z = 5$$

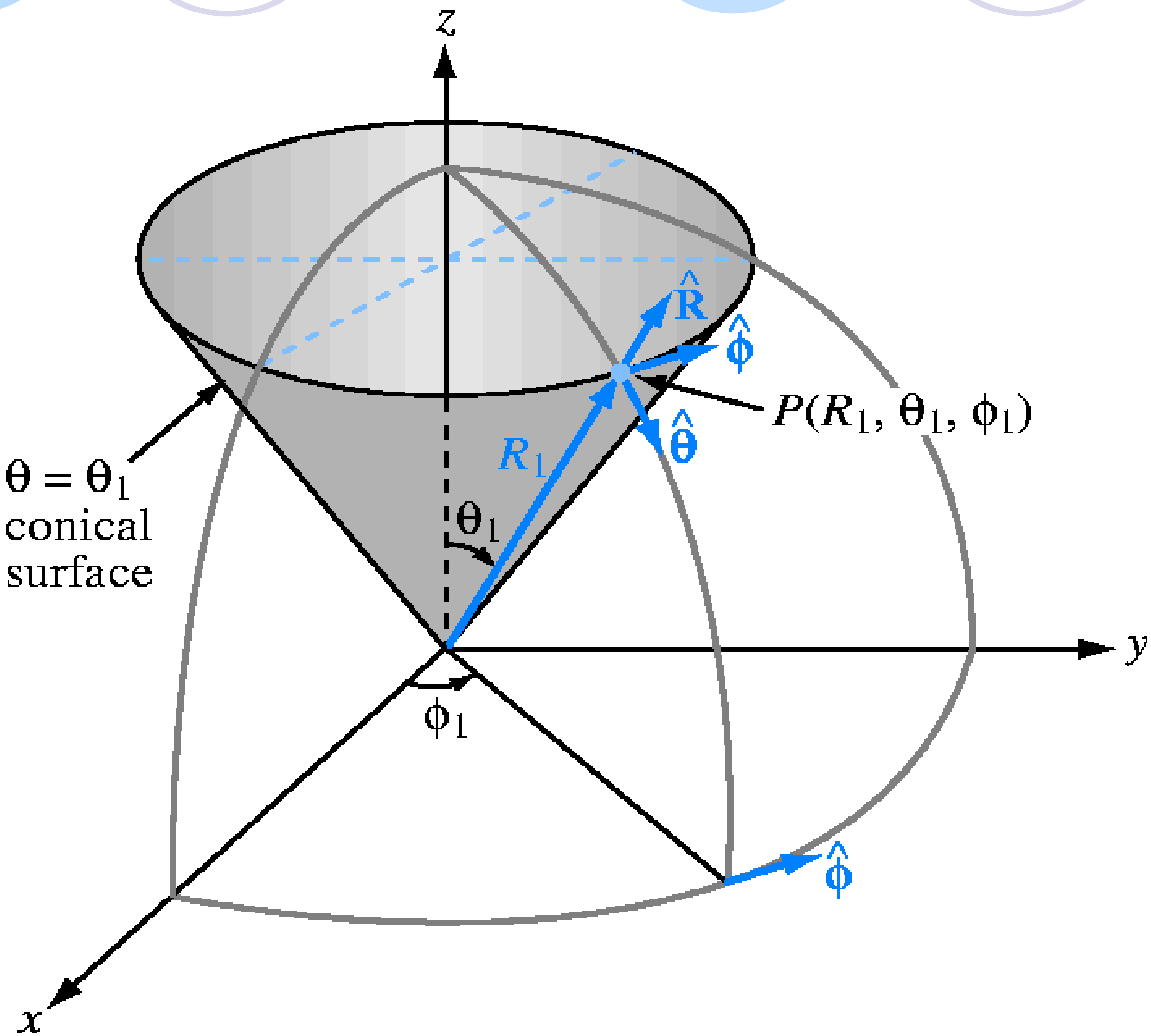
- So a point P (3, 4, 5) in Cartesian coordinate is the same as P (5, 0.927,5) in cylindrical coordinate)

Spherical Coordinates (r, θ, ϕ)

- The spherical coordinate system is used dealing with problems having a degree of spherical symmetry.

- Point P represented as (r,θ,φ) where
  - r : the distance from the origin,
  - θ : called the *colatitude* is the angle between z-axis and vector of P,
  - Φ : *azimuthal* angle or the angular displacement from x-axis (the same azimuthal angle in cylindrical coordinates).

Spherical Coordinates



Spherical Coordinates (r,θ,φ)

- The range of the variables are
  $0 \le r < \infty$, $0 \le \theta < \pi$, $0 < \varphi < 2\pi$
- A vector **A** in spherical coordinates written as
  $(A_r, A_\theta, A_\varphi)$ or $A_r a_r + A_\theta a_\theta + A_\varphi a_\varphi$
- The magnitude of A is

$$|\vec{A}| = \sqrt{{A_r}^2 + {A_\phi}^2 + {A_\theta}^2}$$

Relation to Cartesian coordinates system

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \frac{(\sqrt{x^2 + y^2})}{z}$$

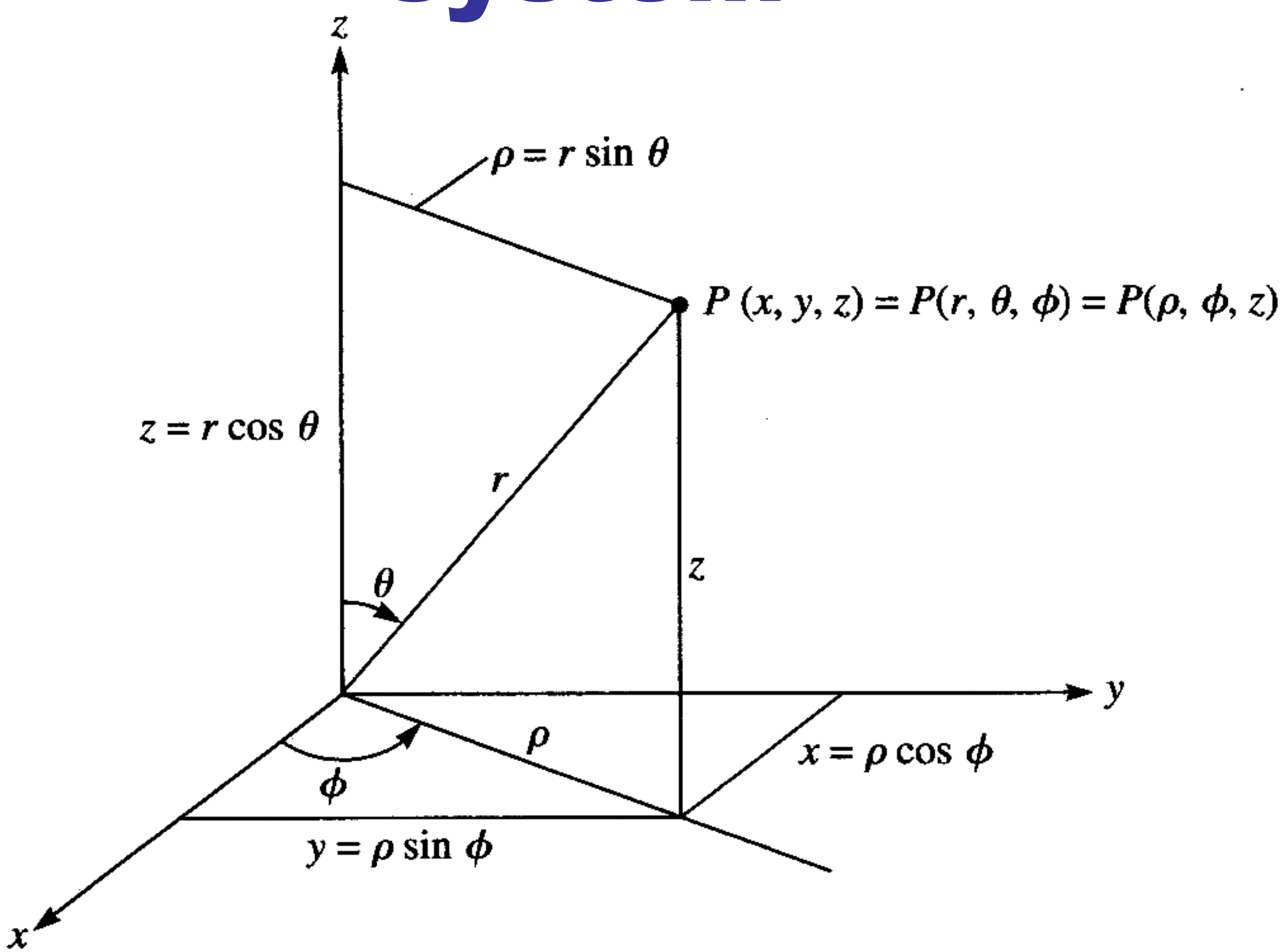
$$\phi = \tan^{-1} \frac{y}{x}$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Relationship between cylinder and spherical coordinate system



Point transformation

Point transformation between cylinder and spherical coordinate is given by

$$r = \sqrt{\rho^2 + z^2} \qquad \theta = \tan^{-1}\frac{\rho}{z} \qquad \phi = \phi$$

or

$$\rho = r\sin\theta \qquad z = r\cos\theta \qquad \phi = \phi$$

Example

Express vector **B** = $\frac{10}{r}\mathbf{a}_r + r\cos\theta\,\mathbf{a}_\theta + \mathbf{a}_\phi$

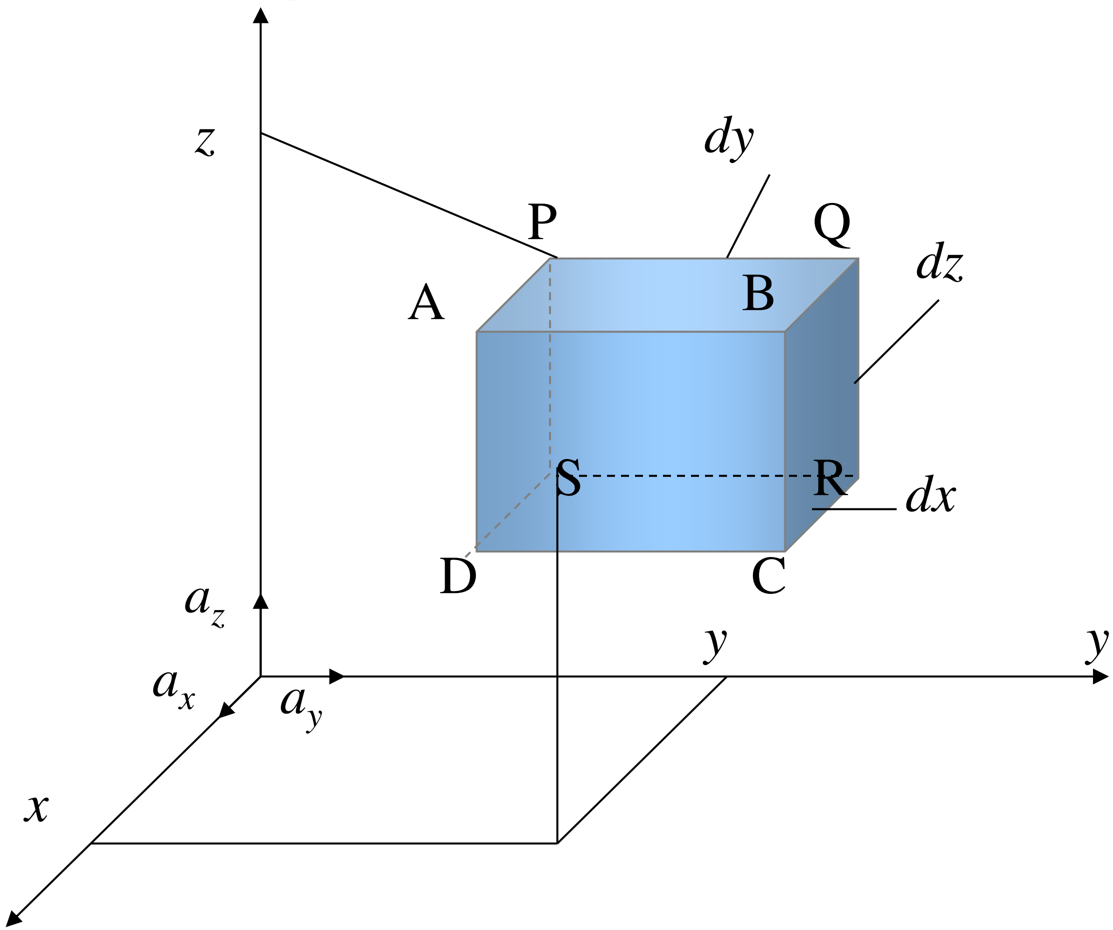
in Cartesian and cylindrical coordinates. Find **B** at (-3, 4 0) and at (5, π/2, -2)

Differential Elements

- In vector calculus the *differential elements* in length, area and volume are useful.

- They are defined in the Cartesian, cylindrical and spherical coordinate

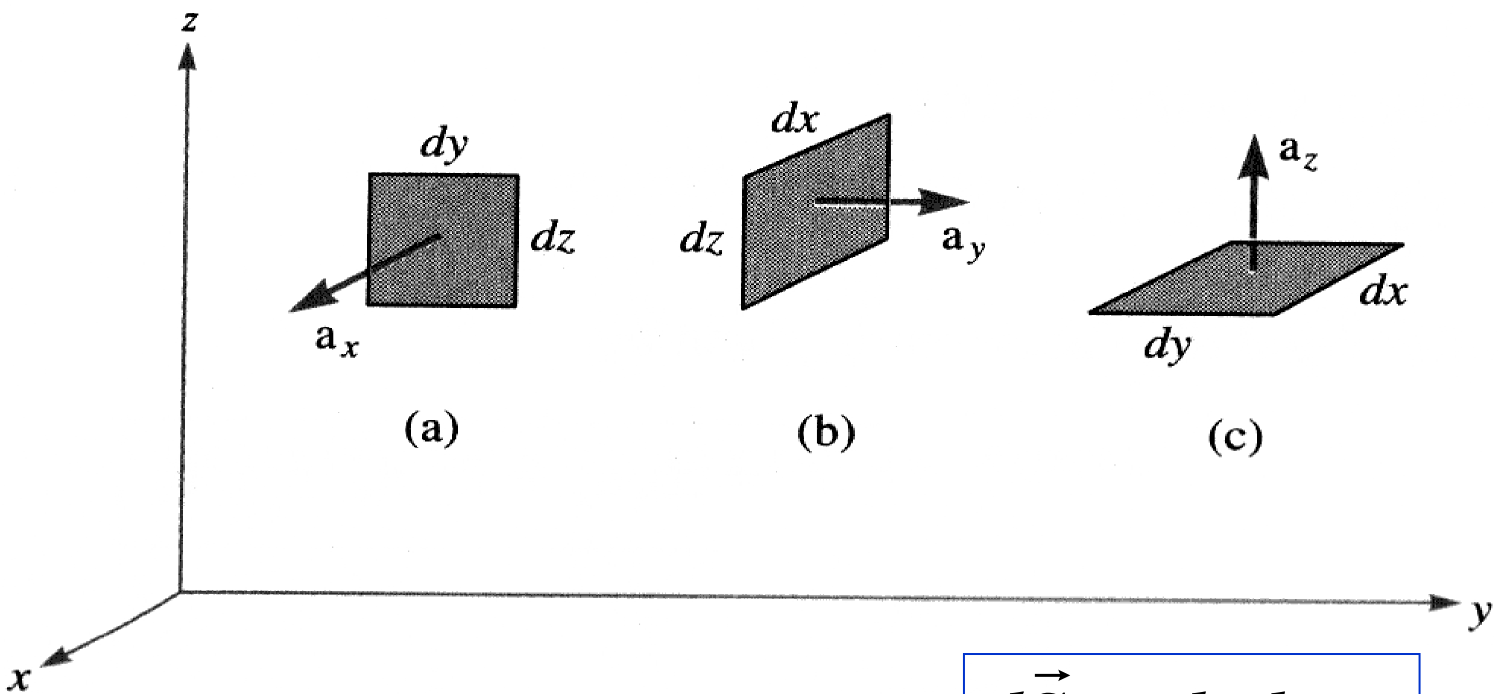
Cartesian Coordinates



Differential displacement :

$$d\vec{l} = dx a_x + dy a_y + dz a_z$$

Cartesian Coordinates



Differential normal area:

$$d\vec{S} = dydz\mathbf{a}_x$$

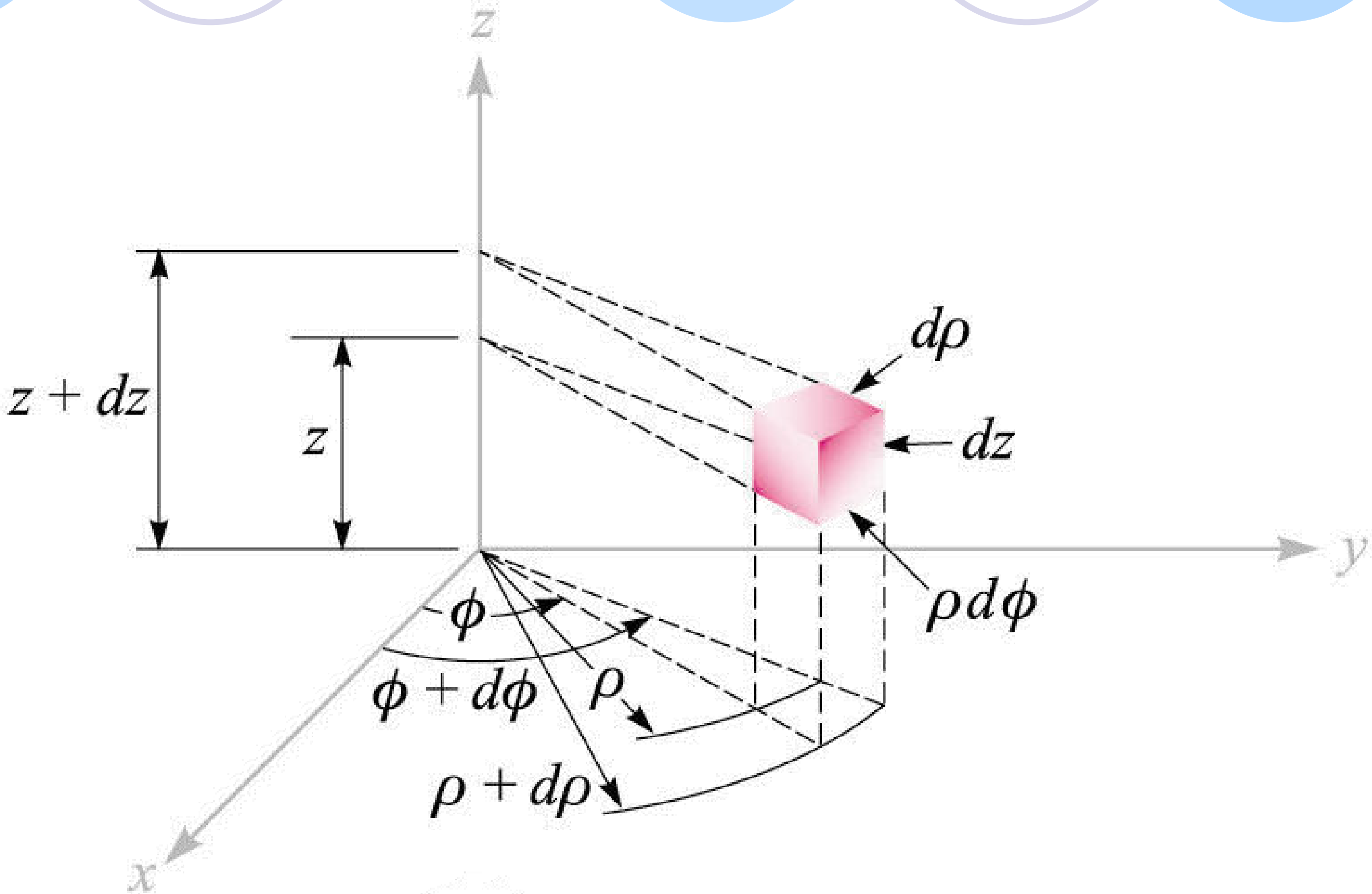
$$d\vec{S} = dxdz\mathbf{a}_y$$

$$d\vec{S} = dxdy\mathbf{a}_z$$

Cartesian Coordinates

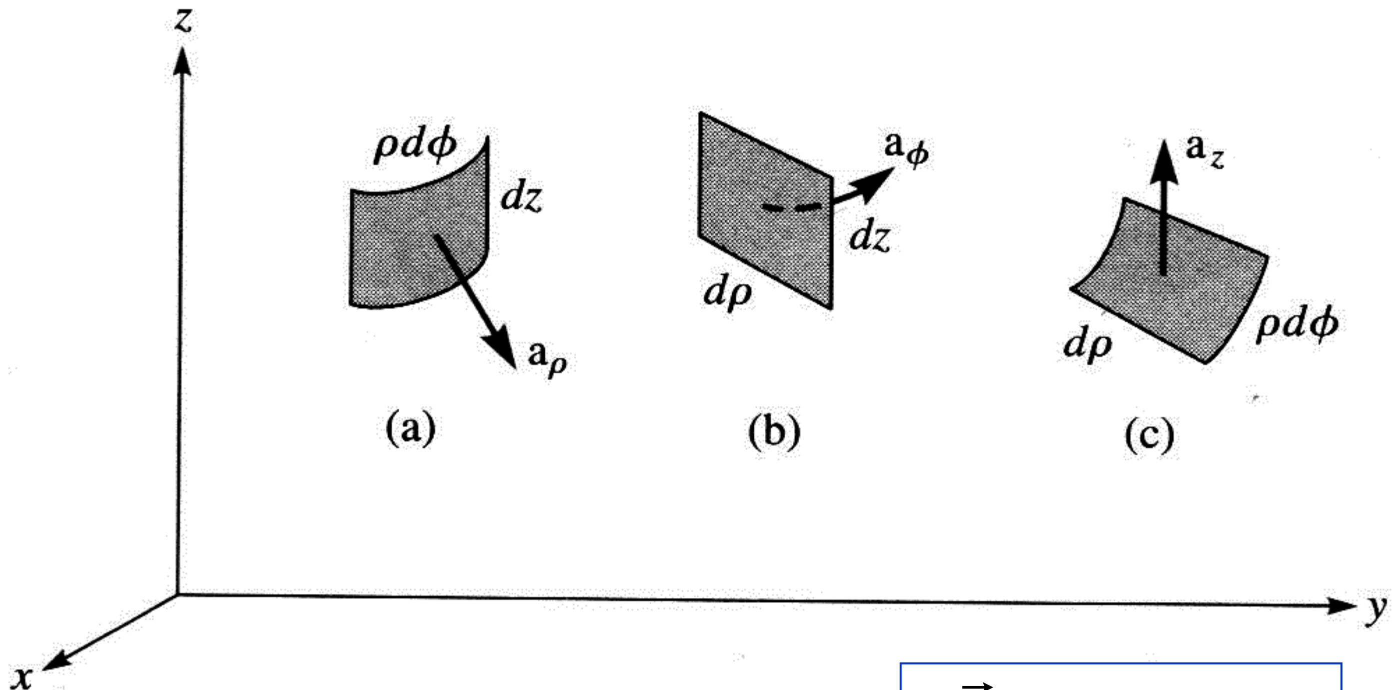
Differential displacement	$d\vec{l} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$
Differential normal area	$d\vec{S} = dydz\mathbf{a}_x$ $d\vec{S} = dxdz\mathbf{a}_y$ $d\vec{S} = dxdy\mathbf{a}_z$
Differential volume	$dv = dxdydz$

Cylindrical Coordinates



Differential displacement : $\vec{dl} = d\rho \mathbf{a}_\rho + \rho d\phi \mathbf{a}_\phi + dz \mathbf{a}_z$

Cylindrical Coordinates



Differential normal area:

$$d\vec{S} = \rho d\phi dz a_{\rho}$$

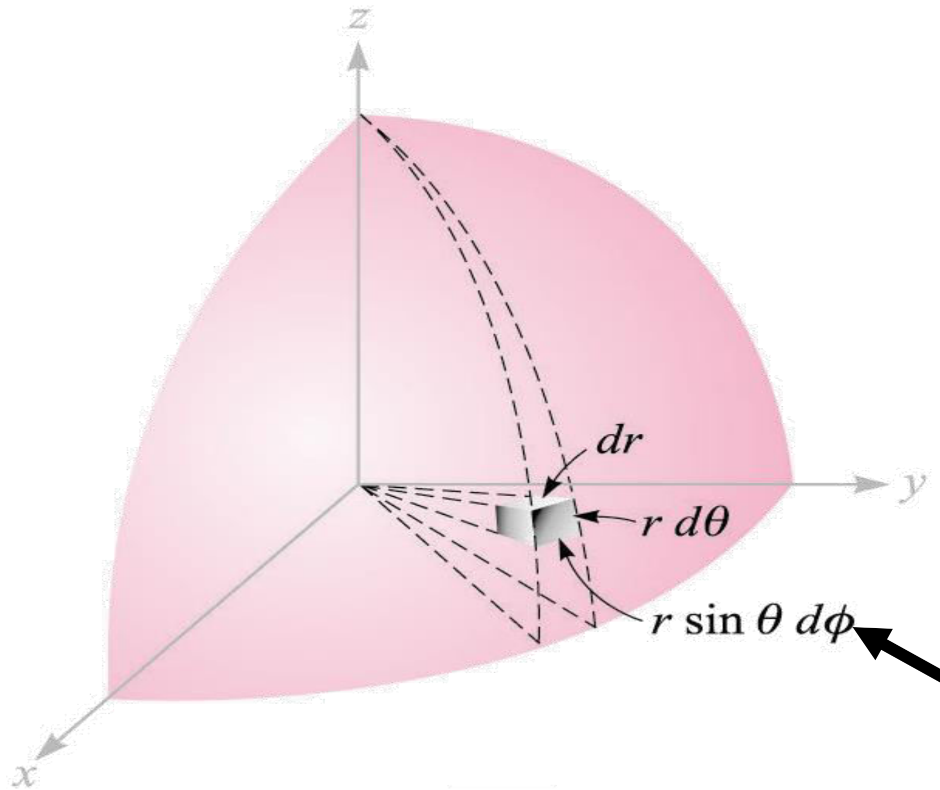
$$d\vec{S} = d\rho dz a_{\phi}$$

$$d\vec{S} = \rho d\phi d\rho a_{z}$$

Cylindrical Coordinates

Differential displacement	$d\vec{l} = d\rho a_\rho + \rho d\phi a_\phi + dz a_z$
Differential normal area	$d\vec{S} = \rho d\phi dz a_\rho$ $d\vec{S} = d\rho dz a_\phi$ $d\vec{S} = \rho d\phi d\rho a_z$
Differential volume	$dv = \rho d\rho d\phi dz$

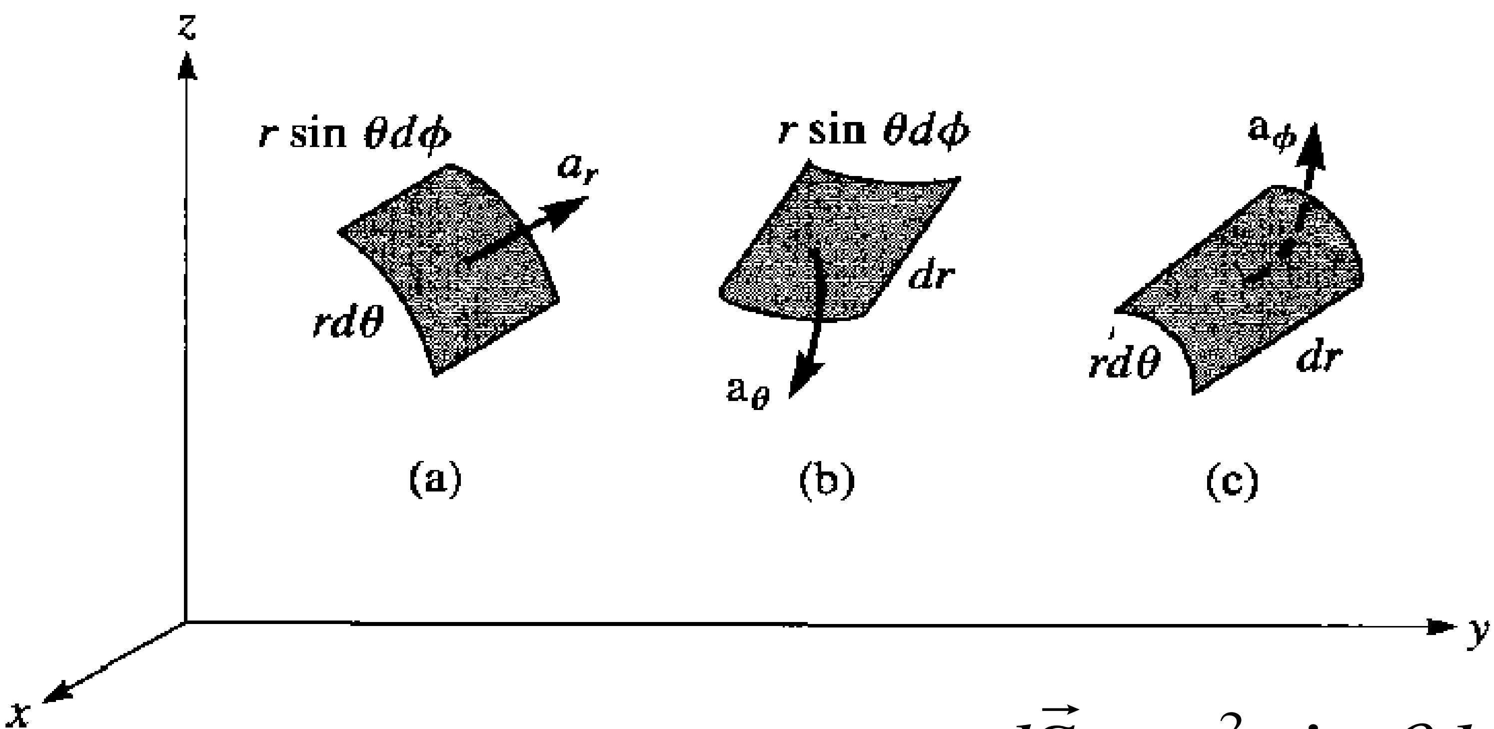
Spherical Coordinates



$\rho d\phi = r\sin\theta d\phi$

Differential displacement : $d\vec{l} = dra_r + rd\theta a_\theta + r\sin\theta d\phi a_\phi$

Spherical Coordinates



Differential normal area:

$$d\vec{S} = r^2 \sin\theta d\theta d\phi \mathbf{a}_r$$

$$d\vec{S} = r \sin\theta dr d\phi \mathbf{a}_\theta$$

$$d\vec{S} = r dr d\theta \mathbf{a}_\phi$$

Spherical Coordinates

Differential displacement	$d\vec{l} = dr a_r + r d\theta a_\theta + r \sin\theta d\phi a_\phi$
Differential normal area	$d\vec{S} = r^2 \sin\theta d\theta d\phi a_r$ $d\vec{S} = r \sin\theta dr d\phi a_\theta$ $d\vec{S} = r dr d\theta a_\phi$
Differential volume	$dv = r^2 \sin\theta dr d\theta d\phi$

Del Operator

- Written as ∇ is the vector differential operator. Also known as the *gradient operator*. The operator in useful in defining:

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector **A**, written as $\nabla \bullet \mathbf{A}$
3. The curl of a vector **A**, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

Gradient of Scalar

- G is the gradient of V. Thus

$$grad\, V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

- In cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$$

- In spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r} \frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} a_\phi$$

Example :

Given : $U = r^2 z\cos 2\phi$

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{a}_r + \frac{\partial U}{r\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial Z}\mathbf{a}_z$$

$$2rz\cos 2\phi\ \mathbf{a}_r - 2rz\sin 2\phi\ \mathbf{a}_\phi + r^2\cos 2\phi\ \mathbf{a}_z$$

Given : $W = 10R\sin^2\theta\cos\phi$

$$\nabla W = \frac{\partial W}{\partial R}\mathbf{a}_R + \frac{\partial W}{R\partial \theta}\mathbf{a}_\theta + \frac{\partial W}{R\sin\theta\partial \phi}\mathbf{a}_\phi$$

$$= 10\sin^2\theta\cos\phi\ \mathbf{a}_R + 10\sin 2\theta\cos\phi\ \mathbf{a}_\theta - 10\sin\theta\sin\phi\mathbf{a}_\phi$$

Divergence

- In Cartesian coordinates,

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- In cylindrical coordinates,

$$\nabla \bullet \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- In spherical coordinate,

$$\nabla \bullet A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Example

Given : $\mathbf{P} = x^2yz\,\mathbf{a_x} + xz\,\mathbf{a_z}$

$$\nabla.\mathbf{P} = \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z$$

$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$

$$= 2xyz + x$$

Curl of a Vector

- In Cartesian coordinates,

$$\nabla \times A = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- In cylindrical coordinates,

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Curl of a Vector

- **In spherical coordinates,**

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & (r\sin\theta) a_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & r A_\theta & (r\sin\theta) A_\phi \end{vmatrix}$$

Examples on Curl Calculation

Given $\mathbf{A} = e^{xy}\,\mathbf{a_x} + \sin xy\,\mathbf{a_y} + \cos^2(xz)\,\mathbf{a_z}$

$$\nabla \times \mathbf{A} = \begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xy} & \sin xy & \cos^2(xz) \end{vmatrix}$$

$$\nabla \mathbf{x} \mathbf{A} = \mathbf{a_x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a_y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a_z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla \mathbf{x} \mathbf{A} = \mathbf{a_x}(0-0) + \mathbf{a_y}(0 + 2z\cos xz \sin xz) + \mathbf{a_z}\left(y\cos xy - xe^{xy}\right)$$

$$\nabla \mathbf{x} \mathbf{A} = \mathbf{a_y}(z\sin 2xz) + \mathbf{a_z}\left(y\cos xy - xe^{xy}\right)$$

Laplacian of a scalar

- The Laplacian of a scalar field V, written as $\nabla^2 V$ is defined as the divergence of the gradient of V.

- In Cartesian coordinates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian of a scalar

- In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

- In spherical coordinates,

$$\nabla^2 \mathrm{V} = \frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2 \frac{\partial \mathrm{V}}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \mathrm{V}}{\partial \theta}\right) + \frac{1}{\mathrm{r}^2 \sin^2\theta}\frac{\partial^2 \mathrm{V}}{\partial \phi^2}$$