Unit -3 & 4
FEMT

Contents

Basic Equations from Vector Calculus

For a scalar function $\varphi(x,y,z,t)$,

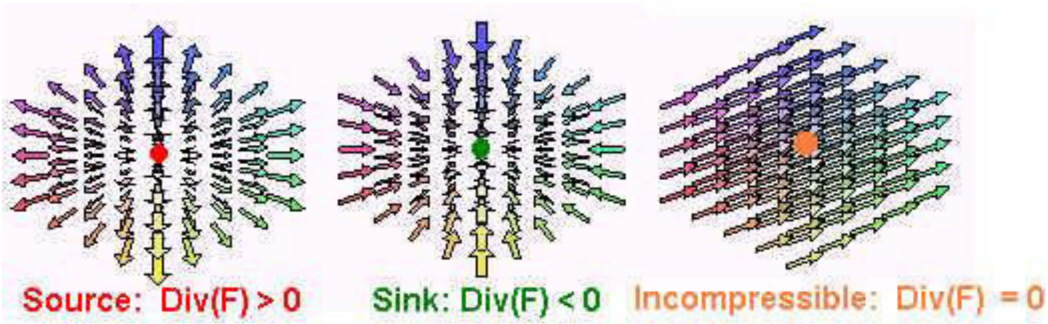
$$\text{gradient}: \quad \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$

Gradient is normal to surfaces
φ=constant

For a vector $\vec{F} = (F_1, F_2, F_3)$,

$$\text{divergence}: \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl}: \quad \nabla\wedge\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$



Basic Vector Calculus

$$\nabla \cdot (\vec{F} \wedge \vec{G}) = \vec{G} \cdot \nabla \wedge \vec{F} - \vec{F} \cdot \nabla \wedge \vec{G}$$

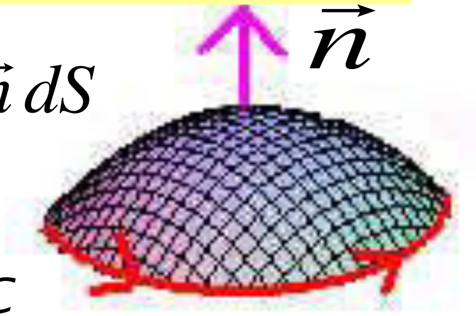
$$\nabla \wedge \nabla \varphi = 0, \quad \nabla \cdot \nabla \wedge \vec{F} = 0$$

$$\nabla \wedge (\nabla \wedge \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

Stokes' Theorem

$$\iint_S \nabla \wedge \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

$$d\vec{S} = \vec{n}\, dS$$



Oriented boundary *C*

Divergence or Gauss' Theorem

$$\iiint_V \nabla \cdot \vec{F}\, dV = \oiint_S \vec{F} \cdot d\vec{S}$$

Closed surface S, volume V, outward pointing normal

What is Electromagnetism?

- The study of Maxwell’s equations, devised in 1863 to represent the relationships between electric and magnetic fields in the presence of electric charges and currents, whether steady or rapidly fluctuating, in a vacuum or in matter.

- The equations represent one of the most elegant and concise way to describe the fundamentals of electricity and magnetism. They pull together in a consistent way earlier results known from the work of Gauss, Faraday, Ampère, Biot, Savart and others.

- Remarkably, Maxwell’s equations are perfectly consistent with the transformations of special relativity.

Maxwell's Equations

Relate Electric and Magnetic fields generated by charge and current distributions.

E = electric field

D = electric displacement

H = magnetic field

B = magnetic flux density

ρ= charge density

j = current density

μ_0 (permeability of free space) = $4\pi\ 10^{-7}$

ε_0 (permittivity of free space) = $8.854\ 10^{-12}$

c (speed of light) = $2.99792458\ 10^8$ m/s

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \wedge \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

In vacuum $\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 c^2 = 1$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

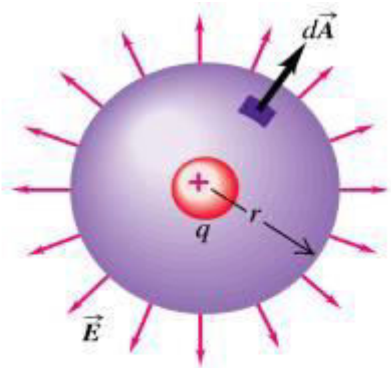
Maxwell's 1st Equation

Equivalent to Gauss' Flux Theorem:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Leftrightarrow \quad \iiint_V \nabla \cdot \vec{E}\, dV = \oiint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho\, dV = \frac{Q}{\varepsilon_0}$$

The flux of electric field out of a closed region is proportional to the total electric charge Q enclosed within the surface.

A point charge *q* generates an electric field



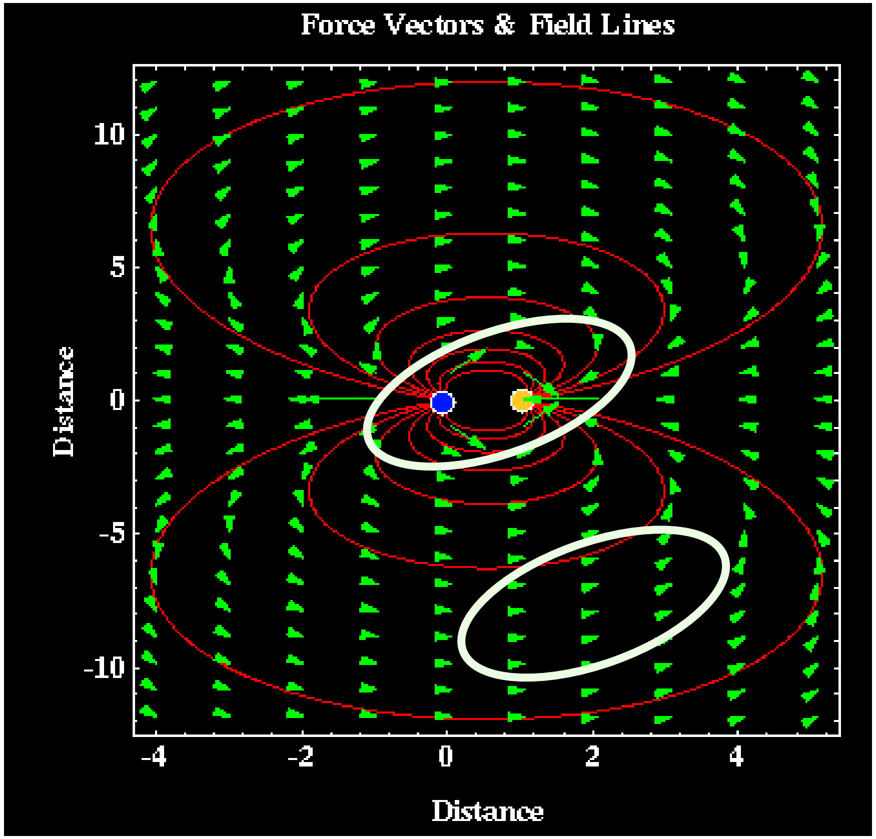
$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$$

$$\iint_{sphere} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0} \iint_{sphere} \frac{dS}{r^2} = \frac{q}{\varepsilon_0}$$

Area integral gives a measure of the net charge enclosed; divergence of the electric field gives the density of the sources.

$$\nabla \cdot \vec{B} = 0$$

Maxwell's 2nd Equation



Gauss' law for magnetism:

$$\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \oiint \vec{B} \cdot d\vec{S} = 0$$

The net magnetic flux out of any closed surface is zero. Surround a magnetic dipole with a closed surface. The magnetic flux directed inward towards the south pole will equal the flux outward from the north pole.

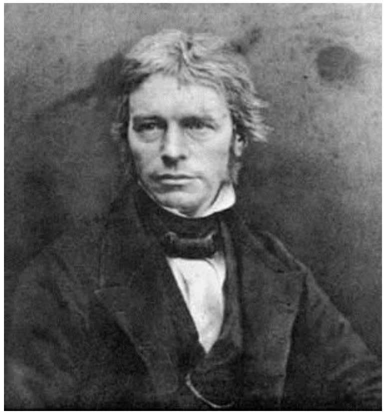
If there were a magnetic monopole source, this would give a non-zero integral.

Gauss' law for magnetism is then a statement that

There are no magnetic monopoles

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell's 3rd Equation

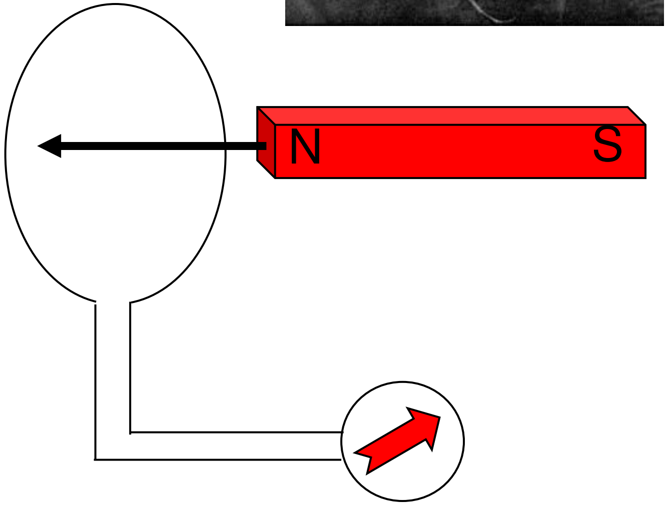


Equivalent to Faraday's Law of Induction:

$$\iint_S \nabla \wedge \vec{E} \cdot d\vec{S} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

$$\Leftrightarrow \quad \oint_C E \cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

(for a fixed circuit *C*)



The electromotive force round a circuit $\varepsilon = \oint \vec{E} \cdot d\vec{l}$ is proportional to the rate of change of flux of magnetic field, $\Phi = \iint \vec{B} \cdot d\vec{S}$ through the circuit.

Faraday's Law is the basis for electric generators. It also forms the basis for inductors and transformers.

$$\nabla \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Maxwell's 4th Equation

Ampère

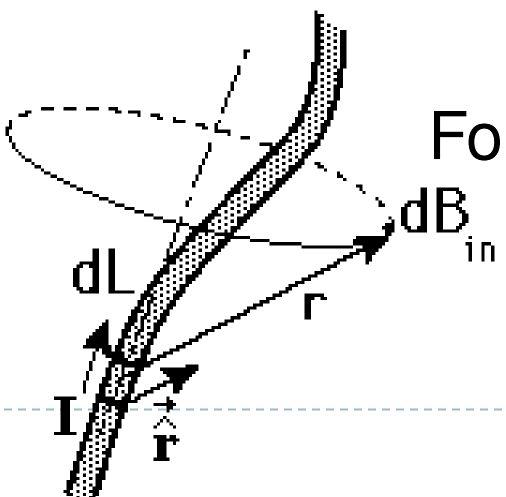
Biot

Originates from Ampère's (Circuital) Law : $\nabla \wedge \vec{B} = \mu_0 \vec{j}$

$$\oint_C \vec{B}\cdot d\vec{l} = \iint_S \nabla \wedge \vec{B}\cdot d\vec{S} = \mu_0 \iint_S \vec{j}\cdot d\vec{S} = \mu_0 I$$

Satisfied by the field for a steady line current (Biot-Savart Law, 1820):

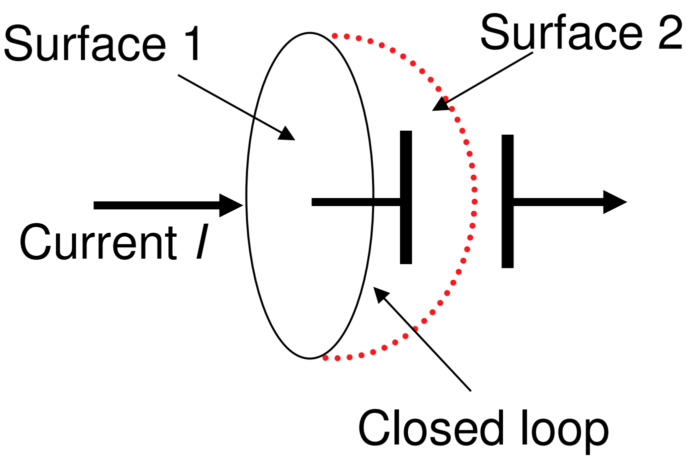
$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l} \wedge \vec{r}}{r^3}$$



For a straightline current $B_\theta = \frac{\mu_0 I}{2\pi r}$

Need for Displacement Current

- Faraday: vary B-field, generate E-field
- Maxwell: varying E-field should then produce a B-field, but not covered by Ampère's Law.



- Apply Ampère to surface 1 (flat disk): line integral of $\boldsymbol{B} = \mu_0 I$
- Applied to surface 2, line integral is zero since no current penetrates the deformed surface.
- In capacitor, $E = \dfrac{Q}{\varepsilon_0 A}$, so $I = \dfrac{dQ}{dt} = \varepsilon_0 A \dfrac{dE}{dt}$
- Displacement current density is $\vec{j}_d = \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

$$\nabla \wedge \vec{B} = \mu_0 \left(\vec{j} + \vec{j}_d \right) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Consistency with Charge Conservation

Charge conservation:

Total current flowing out of a region equals the rate of decrease of charge within the volume.

$$\oiint \vec{j} \cdot d\vec{S} = -\frac{d}{dt} \iiint \rho \, dV$$

$$\Leftrightarrow \quad \iiint \nabla \cdot \vec{j} \, dV = -\iiint \frac{\partial \rho}{\partial t} dV$$

$$\Leftrightarrow \quad \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

From Maxwell's equations:

Take divergence of (modified) Ampère's equation

$$\nabla \cdot \nabla \wedge \overline{B} = \mu_0 \nabla \cdot \vec{j} + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\nabla \cdot \vec{E} \right)$$

$$\Rightarrow \quad 0 = \mu_0 \nabla \cdot \vec{j} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\varepsilon_0} \right)$$

$$\Rightarrow \quad 0 = \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t}$$

Charge conservation is implicit in Maxwell's Equations

Maxwell's Equations in Vacuum

In vacuum

$$\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

Source-free equations:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Source equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \wedge \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

Equivalent integral forms (useful for simple geometries)

$$\oiint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint \rho \, dV$$

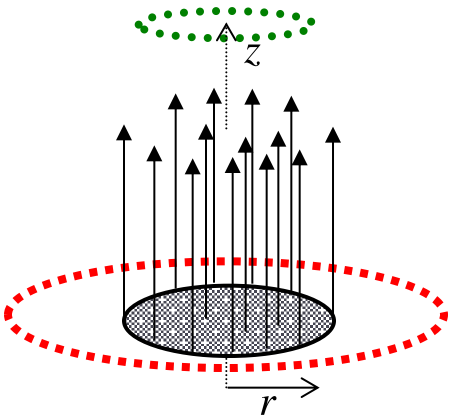
$$\oiint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{j} \cdot d\vec{S} + \frac{1}{c^2} \frac{d}{dt} \iint \vec{E} \cdot d\vec{S}$$

Example: Calculate E from B

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint \vec{B}\cdot dS$$



$$B_z = \begin{cases} B_0 \sin \omega t & r < r_0 \\ 0 & r > r_0 \end{cases}$$

$$r < r_0 \quad 2\pi r E_\theta = -\frac{d}{dt}\pi r^2 B_0 \sin \omega t = -\pi r^2 B_0 \omega \cos \omega t$$

$$\Rightarrow \quad E_\theta = -\frac{1}{2} B_0 \omega r \cos \omega t$$

$$r > r_0 \quad 2\pi r E_\theta = -\frac{d}{dt}\pi r_0^2 B_0 \sin \omega t = -\pi r_0^2 B_0 \omega \cos \omega t$$

$$\Rightarrow \quad E_\theta = -\frac{\omega r_0^2 B_0}{2r} \cos \omega t$$

Also from $\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{dt}$ then gives current density necessary to sustain the fields

Lorentz Force Law

- Supplement to Maxwell's equations, gives force on a charged particle moving in an electromagnetic field:

$$\vec{f} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

- For continuous distributions, have a force density

$$\vec{f}_d = \rho \vec{E} + \vec{j} \wedge \vec{B}$$

- Relativistic equation of motion
  - 4-vector form:

$$F = \frac{dP}{d\tau} \Rightarrow \gamma\left(\frac{\vec{v} \cdot \vec{f}}{c}, \vec{f}\right) = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$$

  - 3-vector component:

$$\frac{d}{dt}\left(m_0 \gamma \vec{v}\right) = \vec{f} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

Motion of charged particles in constant magnetic fields

$$\frac{d}{dt}\left(m_0\gamma\vec{v}\right)=\vec{f}=q\left(\vec{E}+\vec{v}\wedge\vec{B}\right)\quad\rightarrow\quad\frac{d}{dt}\left(m_0\gamma\vec{v}\right)=q\left(\vec{v}\wedge\vec{B}\right)$$

1. Dot product with ***v***:

$$\vec{v}\cdot\frac{d}{dt}\left(\gamma\vec{v}\right)=\frac{q}{m_0}\vec{v}\cdot\vec{v}\wedge\vec{B}=0$$

But $\left(\gamma\vec{v}\right)^2=c^2\left(\gamma^2-1\right)\quad\Rightarrow\quad\gamma\vec{v}\cdot\frac{d}{dt}\left(\gamma\vec{v}\right)=\gamma\frac{d\gamma}{dt}$

So $\frac{d\gamma}{dt}=0\quad\Rightarrow\quad\gamma$ is constant $\quad\Rightarrow\quad\left|\vec{v}\right|$ is constant

No acceleration with a magnetic field

2. Dot product with ***B***:

$$\vec{B}\cdot\frac{d}{dt}\left(\gamma\vec{v}\right)=\frac{q}{m_0}\vec{B}\cdot\vec{v}\wedge\vec{B}=0$$

$$\Rightarrow\quad\frac{d}{dt}\left(\vec{B}\cdot\vec{v}\right)=0,\quad v_{//}=\text{constant}$$

$\left|\vec{v}\right|$ constant and $\left|\vec{v}_{//}\right|$ constant

$\Rightarrow\quad v_{\perp}$ also constant

Motion in constant magnetic field

$$\frac{d\vec{v}}{dt} = \frac{q}{m_0\gamma}\vec{v} \wedge \vec{B}$$

$$\Rightarrow \quad \frac{v_\perp^2}{\rho} = \frac{q}{m_0\gamma} v_\perp B$$

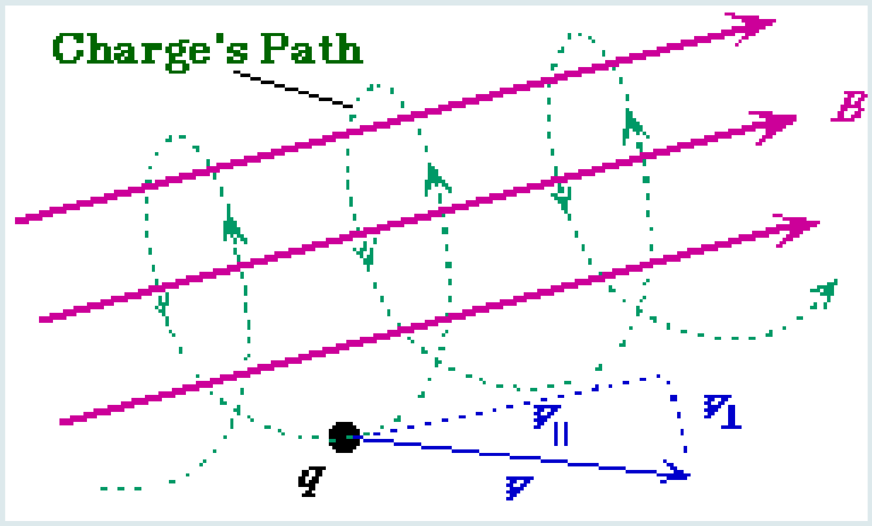
$\Rightarrow$ circular motion with radius $\rho = \dfrac{m_0\gamma v_\perp}{qB}$

at angular frequency $\omega = \dfrac{v_\perp}{\rho} = \dfrac{qB}{m} \quad (m = m_0\gamma)$

Constant magnetic field gives uniform spiral about B with constant energy.

$$B\rho = \frac{m_0\gamma v}{q} = \frac{p}{q}$$

Magnetic rigidity



Motion in constant Electric Field

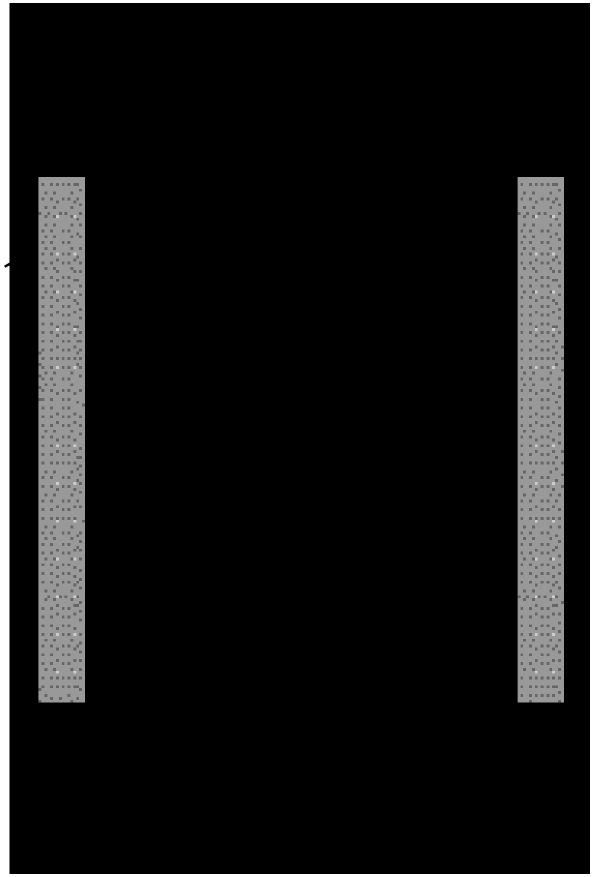
$$\frac{d}{dt}\left(m_0\gamma\vec{v}\right)=\vec{f}=q\left(\vec{E}+\vec{v}\wedge\vec{B}\right)\quad\rightarrow\quad\frac{d}{dt}\left(m_0\gamma\vec{v}\right)=q\vec{E}$$

Solution of $\frac{d}{dt}\left(\gamma\vec{v}\right)=\frac{q}{m_0}\vec{E}$

is $\gamma v=\frac{qE}{m_0}t \quad\Rightarrow\quad \gamma^2=1+\left(\frac{\gamma v}{c}\right)^2 \quad\Rightarrow\quad \gamma =$

$$\frac{dx}{dt}=\frac{\gamma v}{\gamma}\quad\Rightarrow\quad x=\frac{m_0c^2}{qE}\left[\sqrt{1+\left(\frac{qEt}{m_0c}\right)^2}-1\right]$$

$$\approx\frac{1}{2}\frac{qE}{m_0}t^2\quad\text{for}\quad qE<<m_0c$$



Energy gain is qEx

Constant E-field gives uniform acceleration in straight line

Potentials

- Magnetic vector potential:

$$\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \exists\, \vec{A} \text{ such that } \vec{B} = \nabla \wedge \vec{A}$$

- Electric scalar potential:

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\nabla \wedge \vec{A}\right) = -\nabla \wedge \frac{\partial \vec{A}}{\partial t} \quad \Leftrightarrow \quad \nabla \wedge \left(E + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\Leftrightarrow \quad \exists\, \phi \text{ with } \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi, \quad \text{so} \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

- Lorentz Gauge: $\phi \rightarrow \phi + f(t), \quad \vec{A} \rightarrow \vec{A} + \nabla \chi$

Use freedom to set

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

Electromagnetic 4-Vectors

Lorentz Gauge

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t}+\nabla\cdot\vec{A}=0=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right)\cdot\left(\frac{1}{c}\phi,\vec{A}\right)=\nabla_4\cdot\mathrm{A}$$

4-gradient ∇_4 4-potential A

Current 4-vector

$$\vec{j}=\rho\vec{v}$$

$$\Rightarrow \quad J=\rho_0 V=\rho_0\gamma(c,\vec{v})=(\rho c,\vec{j}) \quad \text{where } \rho=\rho_0\gamma$$

Continuity equation

$$\nabla_4\cdot J=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right)\cdot\left(c\rho,\vec{j}\right)=\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{j}=0$$

Charge-current transformations

$$j_x'=\gamma\left(j_x-\rho v\right), \quad \rho'=\gamma\left(\rho-\frac{v\,j_x}{c^2}\right)$$

Relativistic Transformations

- 4-potential vector: $A = \left(\frac{1}{c}\phi, \vec{A}\right)$

- Lorentz transformation $\begin{bmatrix} \phi'/c \\ A'_x \\ A'_y \\ A'_z \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{bmatrix}$

- Fields:

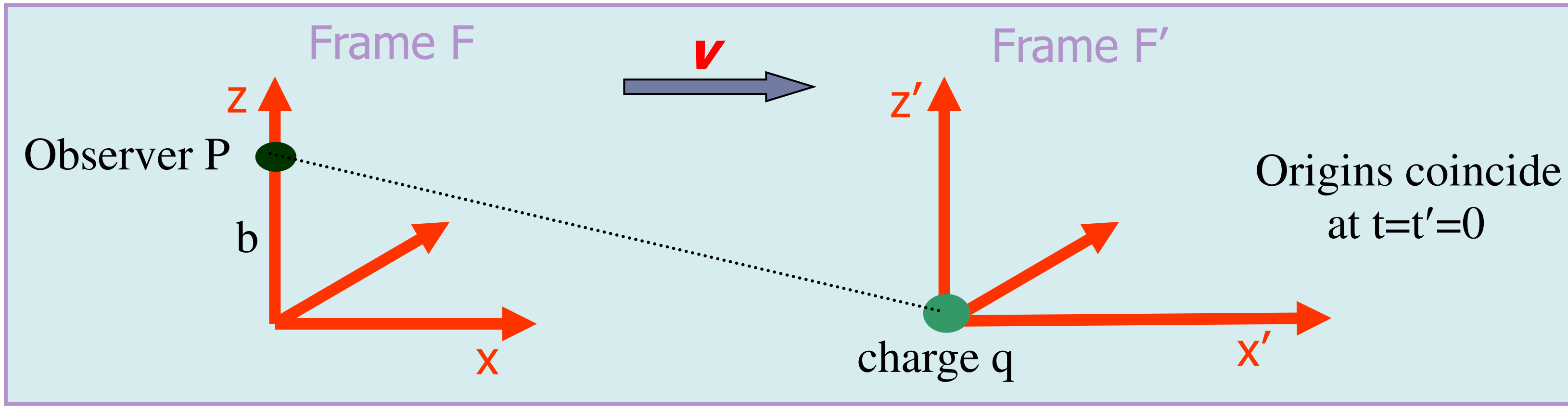
$$\vec{B}' = \nabla' \wedge \vec{A}' \quad \Rightarrow \quad B'_z = \frac{\partial A'_y}{\partial x'} - \frac{\partial A'_x}{\partial y'} \quad \text{and} \quad y' = y, \quad x' = \gamma(x - vt)$$

$$\vec{E}' = -\nabla' \phi' - \frac{\partial \vec{A}'}{\partial t'} \quad \Rightarrow \quad E'_z = -\frac{\partial \phi'}{\partial z'} - \frac{\partial A'_z}{\partial t'} \quad \text{and} \quad z' = z, t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\vec{B}'_{//} = \vec{B}_{//} \quad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v} \wedge \vec{E}}{c^2}\right) \quad \vec{E}'_{//} = \vec{E}_{//} \quad \vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \vec{v} \wedge \vec{B}\right)$$

Example: Electromagnetic Field of a Single Particle

- Charged particle moving along x-axis of Frame F



- P has $0 = x_P = \gamma(x'_P + vt')$ so $x'_P = -vt'$

$$\vec{x}'_P = (-vt', 0, b), \text{ so } \left|\vec{x}'_p\right| = r' = \sqrt{b^2 + v^2 t'^2}, \quad t' = \gamma\left(t - \frac{v x_p}{c^2}\right) = \gamma t$$

- In F′, fields are only electrostatic (***B=0***), given by

$$\vec{E}' = \frac{q}{r'^3}\vec{x}'_P \quad \Rightarrow \quad E'_x = -\frac{qvt'}{r'^3}, \; E'_y = 0, \; E'_z = \frac{qb}{r'^3}$$

$$E'_x = -\frac{qvt'}{r'^3},\quad E'_y = 0,\quad E'_z = \frac{qb}{r'^3}$$

Transform to laboratory frame *F*:

$$\vec{B}'_{//} = \vec{B}_{//} \qquad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}\wedge\vec{E}}{c^2}\right)$$

$$\vec{E}'_{//} = \vec{E}_{//} \qquad \vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \vec{v}\wedge\vec{B}\right)$$

$$E_x = E'_x = -\frac{q\gamma vt}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}}$$

$$E_y = 0$$

$$E_z = \gamma E'_z = \frac{q\gamma b}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}}$$

$$B_y = -\frac{\gamma v}{c^2}E'_z = -\frac{\beta}{c}E_z$$

$$B_x = B_z = 0$$

At non-relativistic energies, $\gamma \approx 1$, restoring the Biot-Savart law:

$$\vec{B} \propto q\frac{\vec{v}\wedge\vec{r}}{r^3}$$

Electromagnetic Energy

- Rate of doing work on unit volume of a system is

$$-\vec{v}\cdot\vec{f}_d = -\vec{v}\cdot\left(\rho\vec{E}+\vec{j}\wedge\vec{B}\right) = -\rho\vec{v}\cdot\vec{E} = -\vec{j}\cdot\vec{E}$$

- Substitute for ***j*** from Maxwell's equations and re-arrange into the form

$$-\vec{j}\cdot\vec{E} = -\left(\nabla\wedge\vec{H}-\frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{E} = \nabla\cdot\vec{E}\wedge\vec{H}-\vec{H}\cdot\nabla\wedge\vec{E}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$$

$$= \nabla\cdot S+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \quad \text{where} \quad \vec{S}=\vec{E}\wedge\vec{H}$$

$$= \nabla\cdot S+\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)$$

Poynting vector

$$-\vec{j}\cdot\vec{E}=\frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\vec{B}\cdot\vec{H}+\vec{E}\cdot\vec{D}\right)\right\}+\nabla\cdot\left(\vec{E}\wedge\vec{H}\right)$$

Integrated over a volume, have energy conservation law: rate of doing work on system equals rate of increase of stored electromagnetic energy+ rate of energy flow across boundary.

$$\frac{dW}{dt}=\frac{d}{dt}\iiint\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)dV+\iint\vec{E}\wedge\vec{H}\cdot d\vec{S}$$

electric + magnetic energy densities of the fields

Poynting vector gives flux of e/m energy across boundaries

Review of Waves

- 1D wave equation is $\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$ with general solution

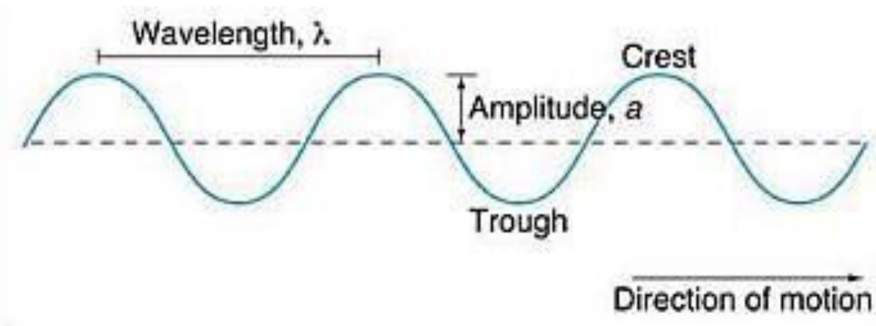
$$u(x,t) = f(vt - x) + g(vt + x)$$

- Simple plane wave:

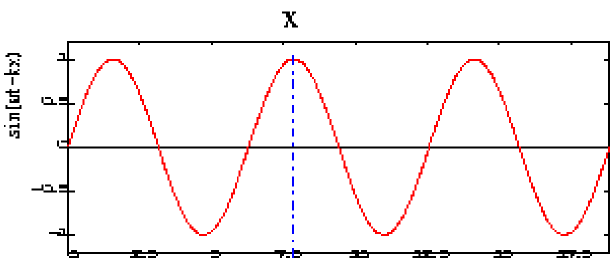
$$\text{1D}: \quad \sin(\omega t - kx) \quad \text{3D}: \quad \sin\left(\omega t - \vec{k}\cdot\vec{x}\right)$$

Wavelength is $\lambda = \frac{2\pi}{|\vec{k}|}$

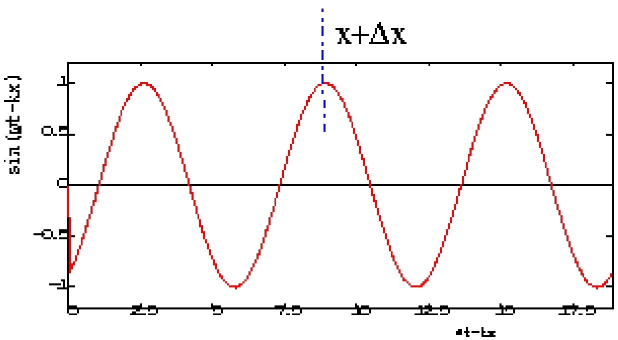
Frequency is $\nu = \frac{\omega}{2\pi}$



Phase and group velocities



Time t

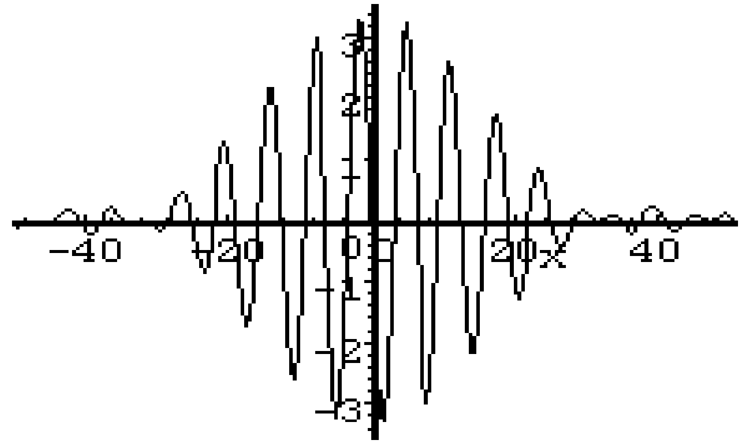


Time t+Δt

Plane wave $\sin(\omega t - kx)$ has constant phase $\omega t - kx = \pi/2$ at peaks

$$\omega \Delta t - k \Delta x = 0$$

$$\Leftrightarrow \quad v_p = \frac{\Delta x}{\Delta t} = \frac{\omega}{k}$$

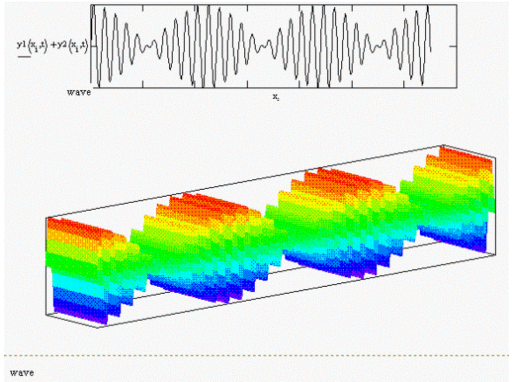


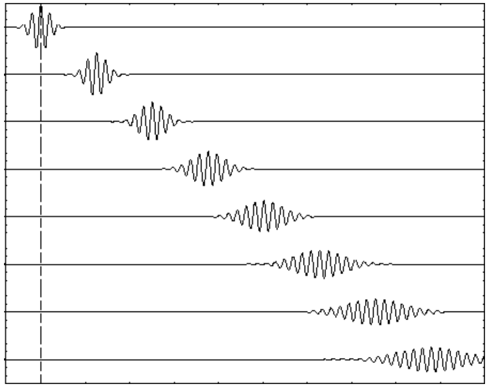
$$\int_{-\infty}^{\infty} A(k) e^{i[\omega(k)t - kx]} dk$$

Superposition of plane waves. While shape is relatively undistorted, pulse travels with the group velocity

$$v_g = \frac{d\omega}{dk}$$

Wave packet structure





- Phase velocities of individual plane waves making up the wave packet are different,
- The wave packet will then disperse with time

Electromagnetic waves

- Maxwell's equations predict the existence of electromagnetic waves, later discovered by Hertz.
- No charges, no currents:

$$\nabla \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \left(\nabla \wedge \vec{E}\right) = -\nabla \wedge \frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left(\nabla \wedge \vec{B}\right)$$

$$= -\mu \frac{\partial^2 \vec{D}}{\partial t^2} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \wedge \left(\nabla \wedge \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$= -\nabla^2 \vec{E}$$

3D wave equation :

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Nature of Electromagnetic Waves

- A general plane wave with angular frequency ω travelling in the direction of the wave vector $\boldsymbol{k}$ has the form

$$\vec{E} = \vec{E}_0 \exp[i(\omega t - \vec{k} \cdot \vec{x})] \quad \vec{B} = \vec{B}_0 \exp[i(\omega t - \vec{k} \cdot \vec{x})]$$

- Phase $\omega t - \vec{k} \cdot \vec{x}$ = 2π × number of waves and so is a Lorentz invariant.
- Apply Maxwell's equations

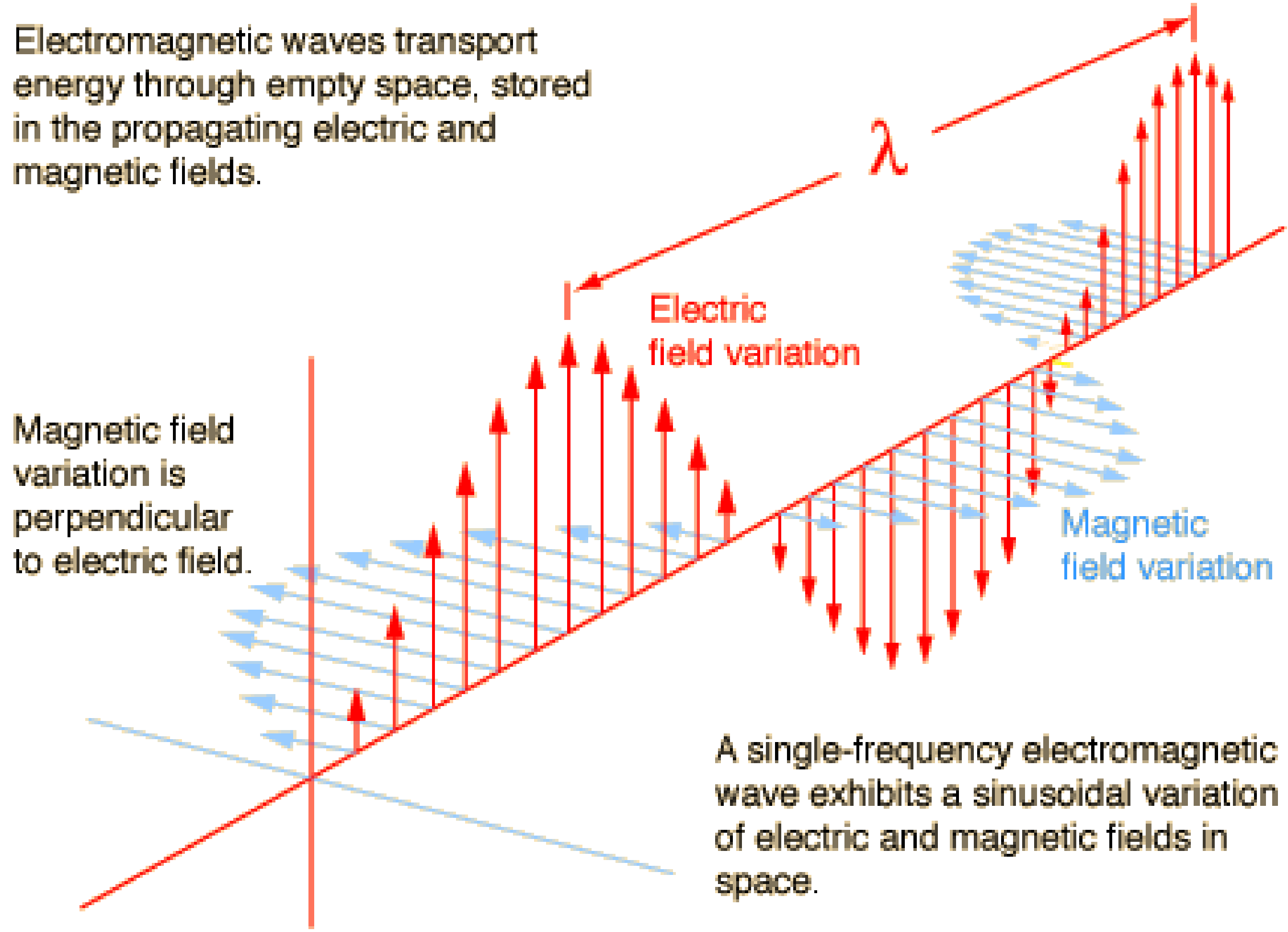
$$\nabla \leftrightarrow -i\vec{k} \qquad \frac{\partial}{\partial t} \leftrightarrow i\omega$$

$$\nabla \cdot \vec{E} = 0 = \nabla \cdot \vec{B} \quad \leftrightarrow \quad \vec{k} \cdot \vec{E} = 0 = \vec{k} \cdot \vec{B}$$

$$\nabla \wedge \vec{E} = -\dot{\vec{B}} \quad \leftrightarrow \quad \vec{k} \wedge \vec{E} = \omega \vec{B}$$

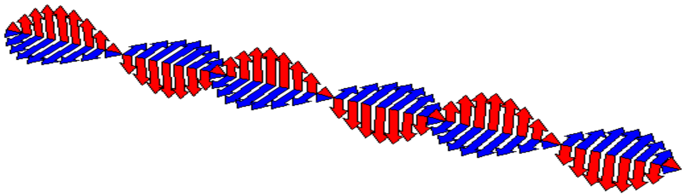
Waves are transverse to the direction of propagation,
$\vec{k}$ and $\vec{E}, \vec{B}$ and are mutually perpendicular

Plane Electromagnetic Wave



Plane Electromagnetic Waves

$$\nabla \wedge \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad \leftrightarrow \quad \vec{k} \wedge \vec{B} = -\frac{\omega}{c^2}\vec{E}$$



Combined with $\vec{k} \wedge \vec{E} = \omega \vec{B}$

deduce that $\dfrac{|\vec{E}|}{|\vec{B}|} = \dfrac{\omega}{k} = \dfrac{kc^2}{\omega}$

$\Rightarrow$ speed of wave in vacuum is $\dfrac{\omega}{|\vec{k}|} = c$

Wavelength $\lambda = \dfrac{2\pi}{|\vec{\mathrm{k}}|}$

Frequency $\nu = \dfrac{\omega}{2\pi}$

Reminder: The fact that $\omega t - \vec{k} \cdot \vec{x}$ is an invariant tells us that

$$\Lambda = \left(\frac{\omega}{c}, \vec{k} \right)$$

is a Lorentz 4-vector, the 4-Frequency vector. Deduce frequency transforms as

$$\omega' = \gamma\left(\omega - \vec{v} \cdot \vec{k}\right) = \omega\sqrt{\frac{c-v}{c+v}}$$

Waves in a Conducting Medium

$$\vec{E} = \vec{E}_0 \exp[i(\omega t - \vec{k}\cdot\vec{x})] \quad \vec{B} = \vec{B}_0 \exp[i(\omega t - \vec{k}\cdot\vec{x})]$$

- (Ohm's Law) For a medium of conductivity σ,

$$\vec{j} = \sigma \vec{E}$$

- Modified Maxwell:

$$\nabla \wedge \vec{H} = \vec{j} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$-i\vec{k} \wedge \vec{H} = \sigma \vec{E} + i\omega\varepsilon \vec{E}$$

conduction current

displacement current

- Put $D = \dfrac{\sigma}{\omega\varepsilon}$

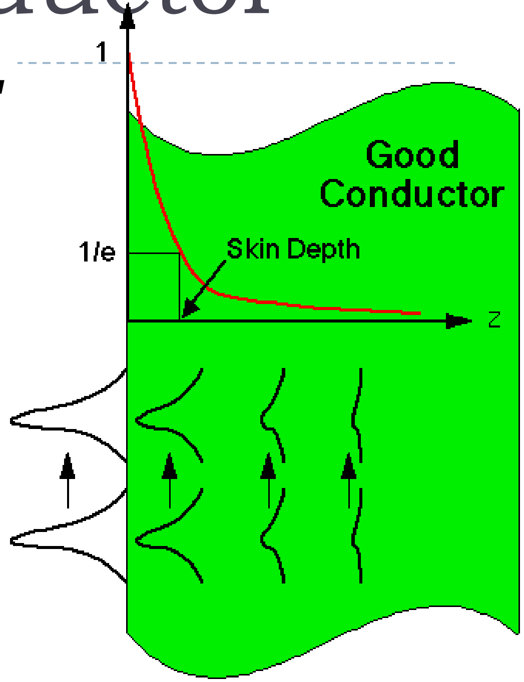
Dissipation factor

Copper: $\sigma = 5.8\times 10^{7}, \varepsilon = \varepsilon_0 \quad \Rightarrow \quad D = 10^{12}$

Teflon: $\sigma = 3\times 10^{-8}, \varepsilon = 2.1\varepsilon_0 \quad \Rightarrow \quad D = 2.57\times 10^{-4}$

Attenuation in a Good Conductor

$$-i\vec{k}\wedge\vec{H}=\sigma\vec{E}+i\omega\varepsilon\vec{E}$$



Combine with $\nabla\wedge\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \quad\Rightarrow\quad \vec{k}\wedge\vec{E}=\omega\mu\vec{H}$

$\Rightarrow \quad \vec{k}\wedge\left(\vec{k}\wedge\vec{E}\right)=\omega\mu\vec{k}\wedge\vec{H}=-\omega\mu(-i\sigma+\omega\varepsilon)\vec{E}$

$\Rightarrow \quad \left(\vec{k}\cdot\vec{E}\right)\vec{k}-k^2\vec{E}=-\omega\mu(-i\sigma+\omega\varepsilon)\vec{E}$

$\Rightarrow \quad k^2=\omega\mu(-i\sigma+\omega\varepsilon)$ since $\vec{k}\cdot\vec{E}=0$

For a good conductor D >> 1, $\sigma>>\omega\varepsilon, \quad k^2\approx-i\omega\mu\sigma \quad\Rightarrow\quad k\approx\sqrt{\dfrac{\omega\mu\sigma}{2}}(1-i)$

Wave form is $\exp\left[i\left(\omega t-\dfrac{x}{\delta}\right)\right]\exp\left(-\dfrac{x}{\delta}\right), \quad k=\dfrac{1}{\delta}(1-i)$ copper.mov water.mov

where $\delta=\sqrt{\dfrac{2}{\omega\mu\sigma}}$ is the skin - depth

Charge Density in a Conducting Material

- Inside a conductor (Ohm's law) $\vec{j} = \sigma \vec{E}$
- Continuity equation is

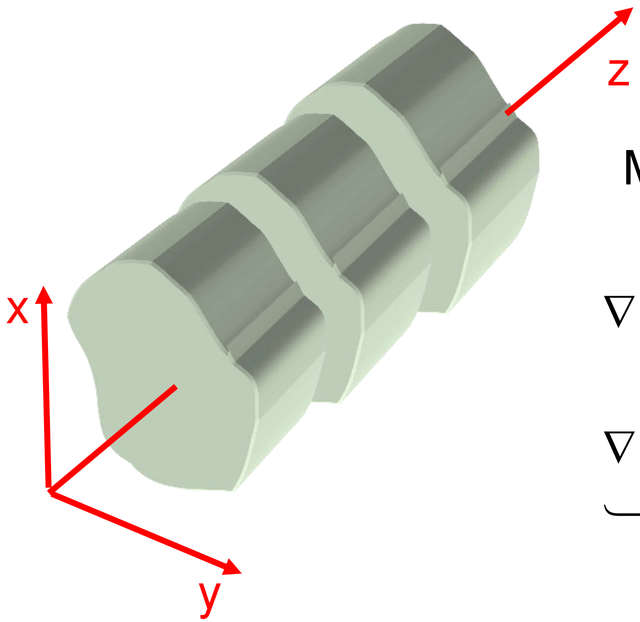
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\Leftrightarrow \quad \frac{\partial \rho}{\partial t} + \sigma \nabla \cdot \vec{E} = 0 = \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon} \rho$$

- Solution is $\rho = \rho_0 e^{-\sigma t/\varepsilon}$

So charge density decays exponentially with time. For a very good conductor, charges flow instantly to the surface to form a surface charge density and (for time varying fields) a surface current. Inside a perfect conductor ($\sigma \to \infty$) ***E=H=0***

Maxwell's Equations in a Uniform Perfectly Conducting Guide



Hollow metallic cylinder with perfectly conducting boundary surfaces

Maxwell's equations with time dependence *exp(iωt)* are:

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega\mu\vec{H}$$

$$\nabla \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t} = i\omega\varepsilon\vec{E}$$

$$\Rightarrow$$

$$\nabla^2\vec{E} = \nabla\left(\nabla\cdot\vec{E}\right) - \nabla\wedge\left(\nabla\wedge\vec{E}\right) = i\omega\mu\nabla\wedge\vec{H} = -\omega^2\varepsilon\mu\,\vec{E}$$

$$\left(\nabla^2 + \omega^2\mu\varepsilon\right)\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$$

Assume $\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{(i\omega t - \gamma z)}$

$\vec{H}(x,y,z,t) = \vec{H}(x,y)e^{(i\omega t - \gamma z)}$

Then $\left[\nabla_t^2 + (\omega^2\varepsilon\mu + \gamma^2)\right]\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$

γ is the propagation constant

Can solve for the fields completely in terms of E_z and H_z

Special cases

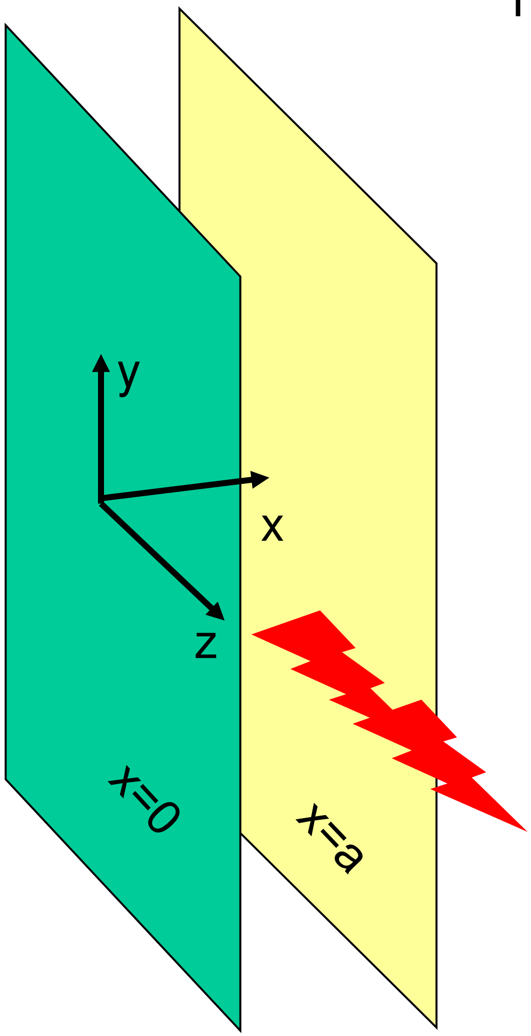
- Transverse magnetic (TM modes):
  - $H_z=0$ everywhere, $E_z=0$ on cylindrical boundary

- Transverse electric (TE modes):
  - $E_z=0$ everywhere, $\frac{\partial H_z}{\partial n}=0$ on cylindrical boundary

- Transverse electromagnetic (TEM modes):
  - $E_z=H_z=0$ everywhere
  - requires

$$\gamma^2 + \omega^2 \varepsilon\mu = 0 \quad \text{or} \quad \gamma = \pm i\omega\sqrt{\varepsilon\mu}$$

A simple model: "Parallel Plate Waveguide"



Transport between two infinite conducting plates (TE_{01} mode):

$$\vec{E} = (0,1,0)E(x)\ e^{(i\omega t - \gamma z)} \quad \text{where } E(x) \text{ satisfies}$$

$$\nabla_t^2 E = \frac{d^2E}{dx^2} = -K^2 E, \quad K^2 = \omega^2 \varepsilon\mu + \gamma^2$$

$$\text{i.e.} \quad E = A \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} Kx$$

To satisfy boundary conditions, $E=0$ on $x=0$ and $x=a$, so

$$E = A \sin Kx, \quad K = K_n = \frac{n\pi}{a}, \quad n \text{ integer}$$

Propagation constant is

$$\gamma = \sqrt{K_n^2 - \omega^2 \varepsilon\mu} = \frac{n\pi}{a}\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} \quad \text{where} \quad \omega_c = \frac{K_n}{\sqrt{\varepsilon\mu}}$$

Cut-off frequency, ω_c

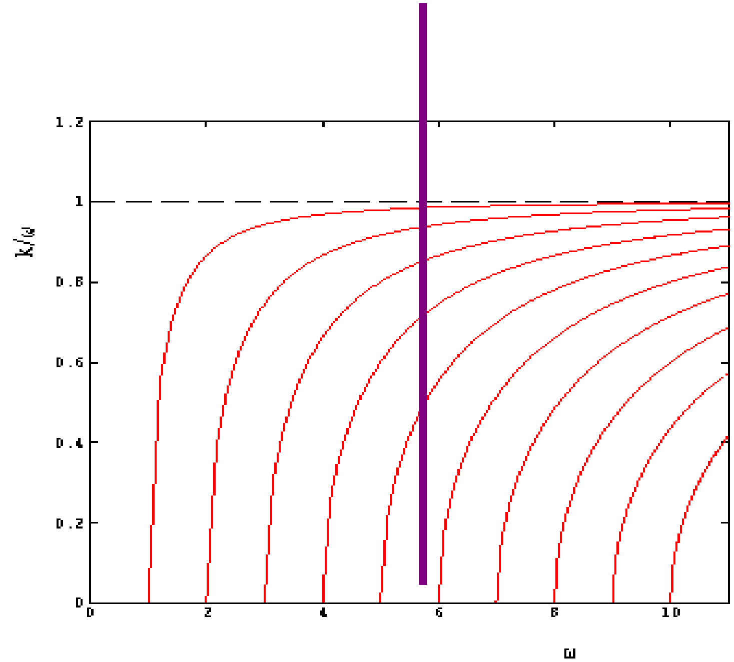
$$\gamma = \frac{n\pi}{a}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}, \quad E = A\sin\frac{n\pi x}{a}\,e^{i\omega t-\gamma z}, \quad \omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}}$$

- $\omega<\omega_c$ gives real solution for γ, so attenuation only. No wave propagates: cut-off modes.
- $\omega>\omega_c$ gives purely imaginary solution for γ, and a wave propagates without attenuation.

$$\gamma = ik, \quad k = \sqrt{\varepsilon\mu}\left(\omega^2-\omega_c^2\right)^{1/2} = \omega\sqrt{\varepsilon\mu}\left(1-\frac{\omega_c^2}{\omega^2}\right)^{1/2}$$

- For a given frequency ω only a finite number of modes can propagate.

$$\omega > \omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}} \quad \Rightarrow \quad n < \frac{a\omega}{\pi}\sqrt{\varepsilon\mu}$$

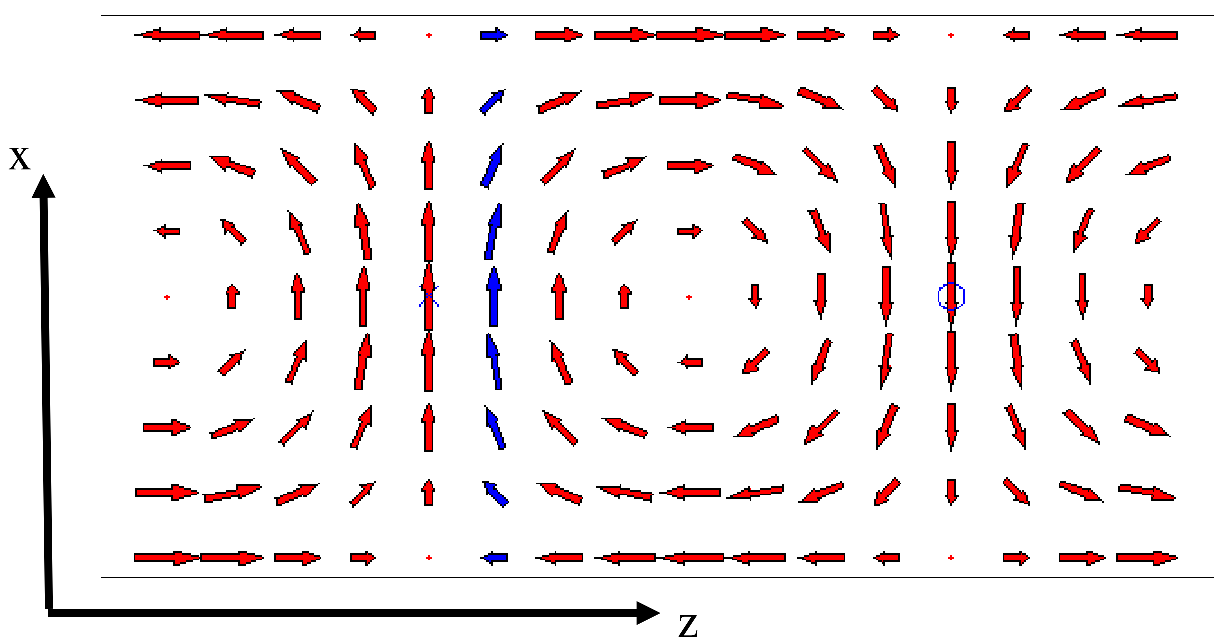


For given frequency, convenient to choose *a* s.t. only *n=1* mode occurs.

Propagated Electromagnetic Fields

From $\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, assuming A is real,

$$\vec{H} = \frac{i}{\omega\mu} \nabla \wedge \vec{E} \quad \Rightarrow \quad \begin{cases} H_x = -\frac{Ak}{\omega\mu} \sin\left(\frac{n\pi x}{a}\right) \cos(\omega t - kz) \\ H_y = 0 \\ H_z = -\frac{A}{\omega\mu} \frac{n\pi}{a} \cos\left(\frac{n\pi x}{a}\right) \sin(\omega t - kz) \end{cases}$$



Phase and group velocities in the simple wave guide

Wave number:

$$k = \sqrt{\varepsilon\mu}\left(\omega^2 - \omega_c^2\right)^{1/2} < \omega\sqrt{\varepsilon\mu}$$

Wavelength:

$$\lambda = \frac{2\pi}{k} > \frac{2\pi}{\omega\sqrt{\varepsilon\mu}}, \quad \text{the free} - \text{space wavelength}$$

Phase velocity:

$$v_p = \frac{\omega}{k} > \frac{1}{\sqrt{\varepsilon\mu}},$$

larger than free - space velocity

Group velocity:

$$k^2 = \varepsilon\mu\left(\omega^2 - \omega_c^2\right) \quad \Rightarrow \quad v_g = \frac{d\omega}{dk} = \frac{k}{\omega\varepsilon\mu} < \frac{1}{\sqrt{\varepsilon\mu}}$$

smaller than free - space velocity

Calculation of Wave Properties

- If a=3 cm, cut-off frequency of lowest order mode is

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2a\sqrt{\varepsilon\mu}} \cong \frac{3\times10^8}{2\times0.03} \cong 5\,\text{GHz}$$

$$\omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}}$$

- At 7 GHz, only the n=1 mode propagates and

$$k = \sqrt{\varepsilon\mu}\left(\omega^2 - \omega_c^2\right)^{1/2} \cong 2\pi\left(7^2 - 5^2\right)^{1/2} \times 10^9 / 3\times10^8 \approx 103\,\text{m}^{-1}$$

$$\lambda = \frac{2\pi}{k} \approx 6\,\text{cm}$$

$$v_p = \frac{\omega}{k} \approx 4.3\times10^8\,\text{ms}^{-1} > c$$

$$v_g = \frac{k}{\omega\varepsilon\mu} = 2.1\times10^8\,\text{ms}^{-1} < c$$

Flow of EM energy along the simple guide

Fields ($\omega>\omega_c$) are:

$$E_x = E_z = 0, \quad E_y = A\sin\frac{n\pi x}{a}\cos(\omega t - kz)$$

$$H_x = -\frac{k}{\omega\mu}E_y, \quad H_y = 0, \quad H_z = -\frac{n\pi}{a\omega\mu}A\cos\frac{n\pi x}{a}\sin(\omega t - kz)$$

Time-averaged energy:

$$\text{Electric energy} \quad W_e = \frac{1}{4}\varepsilon\int_0^a \left|\vec{E}\right|^2 dx = \frac{1}{8}\varepsilon A^2 a$$

$$\text{Magnetic energy} \quad W_m = \frac{1}{4}\mu\int_0^a \left|\vec{H}\right|^2 dx = \frac{1}{8}\mu A^2 a\left\{\left(\frac{n\pi}{a\omega\mu}\right)^2 + \left(\frac{k}{\omega\mu}\right)^2\right\}$$

$$= W_e \quad \text{since} \quad k^2 + \frac{n^2\pi^2}{a^2} = \omega^2\varepsilon\mu$$

Total e/m energy density

$$W = \frac{1}{4}\varepsilon A^2 a$$

Poynting Vector

Poynting vector is $\vec{S} = \vec{E} \wedge \vec{H} = \left(E_y H_z, 0, -E_y H_x\right)$

Time-averaged: $\left\langle \vec{S} \right\rangle = \frac{1}{2}(0,0,1)\frac{kA^2}{\omega\mu}\sin^2\frac{n\pi x}{a}$

Total e/m energy density

$$W = \frac{1}{4}\varepsilon A^2 a$$

Integrate over *x*: $\left\langle S_z \right\rangle = \frac{1}{4}\frac{akA^2}{\omega\mu}$

So energy is transported at a rate: $\frac{\left\langle S_z \right\rangle}{W_e + W_m} = \frac{k}{\omega\varepsilon\mu} = v_g$

Electromagnetic energy is transported down the waveguide with the group velocity