Probability Density Function

- The pdf is defined as the derivative of the cdf:

  fx(x) = d/dx Fx(x)

- It follows that:

$$P(x_1 \le \mathbf{x} \le x_2) = P(\mathbf{x} \le x_2) - P(\mathbf{x} \le x_1)$$
$$= F_{\mathbf{x}}(x_2) - F_{\mathbf{x}}(x_1) = \int_{x_1}^{x_2} f_{\mathbf{x}}(x)\mathrm{d}x.$$

- Note that, for all i, one has pi ≥ 0 and ∑pi = 1.

Cumulative Joint PDF Joint PDF

- Often encountered when dealing with combined experiments or repeated trials of a single experiment.
- Multiple random variables are basically multidimensional functions defined on a sample space of a combined experiment.
- Experiment 1
  - S1 = {x1, x2, …,xm}
- Experiment 2
  - S2 = {y1, y2 , …, yn}
- If we take any one element from S1 and S2
  - 0 <= P(xi, yj) <= 1 (Joint Probability of two or more outcomes)
  - Marginal probabilty distributions
    - Sum all j P(xi, yj) = P(xi)
    - Sum all i P(xi, yj) = P(yi)

Expectation of Random Variables (Statistical averages)

- Statistical averages, or moments, play an important role in the characterization of the random variable.
- The first moment of the probability distribution of a random variable X is called mean value mx or expected value of a random variable X
- The second moment of a probability distribution is mean-square value of X
- Central moments are the moments of the difference between X and mx, and second central moment is the variance of x.
- Variance is equal to the difference between the mean-square value and the square of the mean

$$m_X = \mathbf{E}\{X\} = \int_{-\infty}^{\infty} x p_X(x)\, dx$$

$$\mathbf{E}\{X^2\} = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx$$

$$\text{var}(X) = \mathbf{E}\{(X - m_X)^2\} = \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x)\, dx$$

$$\begin{aligned}\sigma_X^2 &= \mathbf{E}\{X^2 - 2m_X X + m_X^2\} \\ &= \mathbf{E}\{X^2\} - 2m_X \mathbf{E}\{X\} + m_X^2 \\ &= \mathbf{E}\{X^2\} - m_X^2\end{aligned}$$

Contd

- The variance provides a measure of the variable’s “randomness”.
- The mean and variance of a random variable give a partial description of its pdf.

Time Averaging and Ergodicity

- A process where any member of the ensemble exhibits the same statistical behavior as that of the whole ensemble.
- For an ergodic process: To measure various statistical averages, it is sufficient to look at only one realization of the process and find the corresponding time average.
- For a process to be ergodic it must be stationary. The converse is not true.

GAUSSIAN (OR NORMAL) RANDOM VARIABLE (PROCESS)

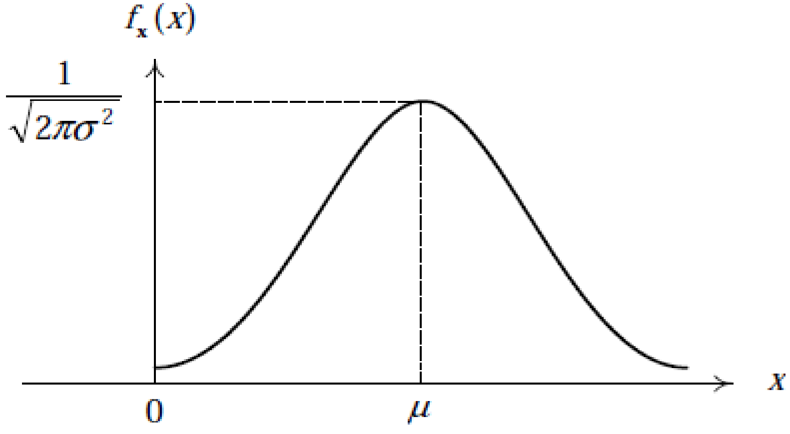
- A continuous random variable whose pdf is:

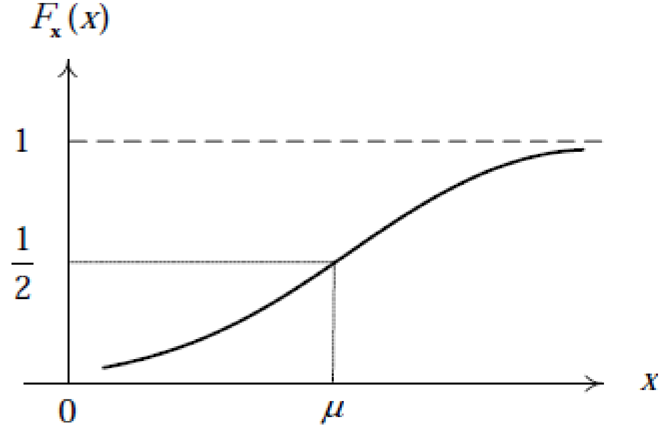
$$f_{\mathbf{x}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

  μ and σ^2 are parameters. Usually denoted as

  N(μ, σ^2) .

- Most important and frequently encountered random variable in communications.





Central Limit Theorem

- CLT provides justification for using Gaussian Process as a model based if
  - The random variables are statistically independent
  - The random variables have probability with same mean and variance

CLT

- The central limit theorem states that
  - “The probability distribution of Vn approaches a normalized Gaussian Distribution N(0, 1) in the limit as the number of random variables approach infinity”
- At times when N is finite it may provide a poor approximation of for the actual probability distribution

Autocorrelation

Autocorrelation of Energy Signals

- Correlation is a matching process; *autocorrelation* refers to the matching of a signal with a delayed version of itself
- The autocorrelation function of a real-valued energy signal $x(t)$ is defined as:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt \qquad \text{for } -\infty < \tau < \infty$$

- The autocorrelation function $R_x(\tau)$ provides a measure of how closely the signal matches a copy of itself as the copy is shifted τ units in time.
- $R_x(\tau)$ is not a function of time; it is only a function of the time difference τ between the waveform and its shifted copy.

AUTOCORRELATION

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0) \text{ for all } \tau$$

$$R_x(\tau) \leftrightarrow \psi_x(f)$$

$$R_x(0) = \int_{-\infty}^{\infty} x^2(t)\, dt$$

- symmetrical in τ about zero
- maximum value occurs at the origin
- autocorrelation and ESD form a Fourier transform pair, as designated by the double-headed arrows
- value at the origin is equal to the energy of the signal

AUTOCORRELATION OF A PERIODIC (POWER) SIGNAL

- The autocorrelation function of a real-valued power signal *x*(*t*) is defined as:

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt \qquad \text{for } -\infty < \tau < \infty$$

- When the power signal *x*(*t*) is periodic with period T_0, the autocorrelation function can be expressed as:

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x(t+\tau)\,dt \qquad \text{for } -\infty < \tau < \infty$$

Autocorrelation of Power Signals

The autocorrelation function of a real-valued *periodic* signal has properties similar to those of an energy signal:

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \le R_x(0) \quad \text{for all } \tau$$

$$R_x(\tau) \leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt$$

- symmetrical in τ about zero
- maximum value occurs at the origin
- autocorrelation and PSD form a Fourier transform pair, as designated by the double-headed arrows
- value at the origin is equal to the average power of the signal

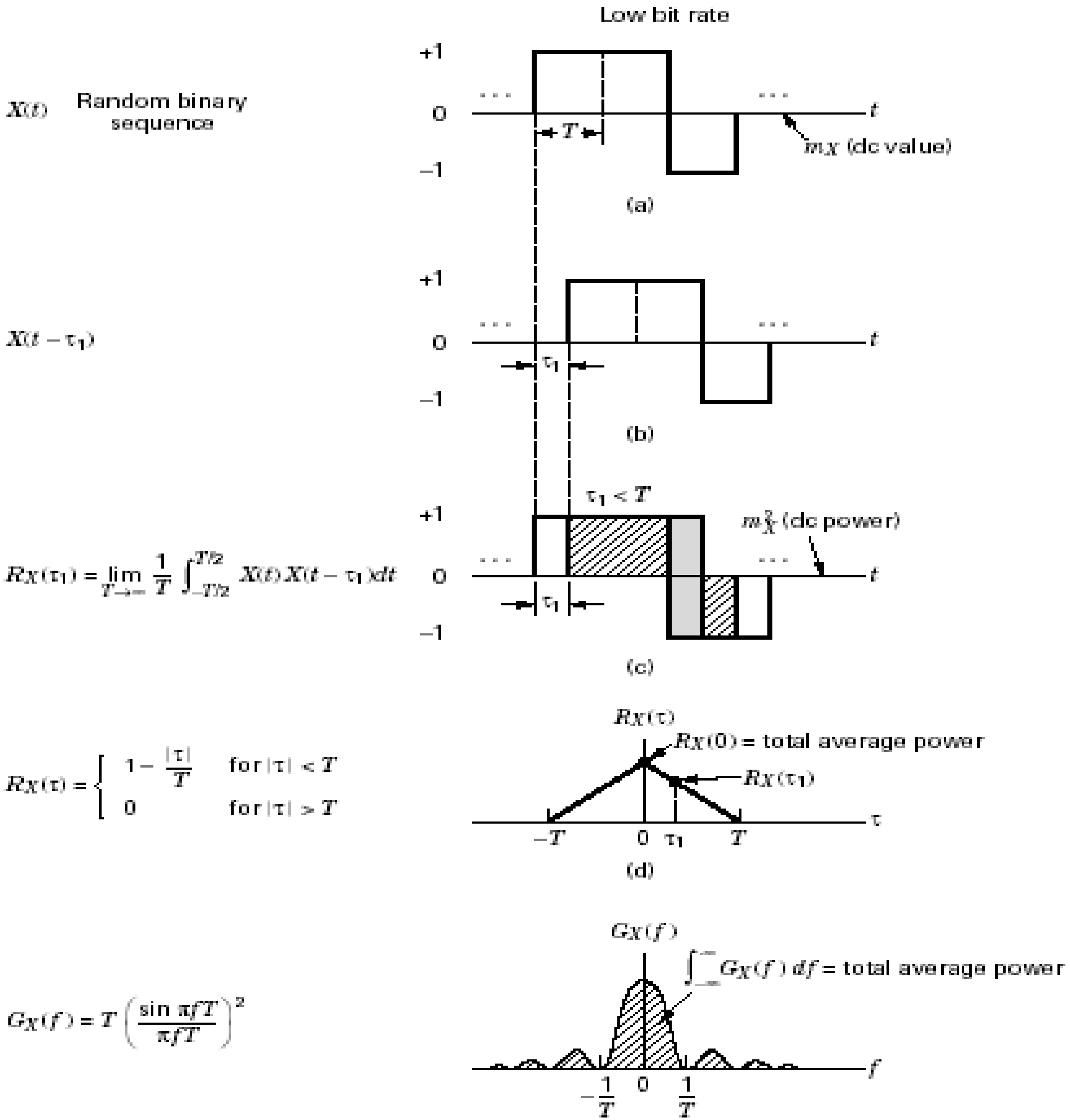


Figure 1.6 Autocorrelation and power spectral density.

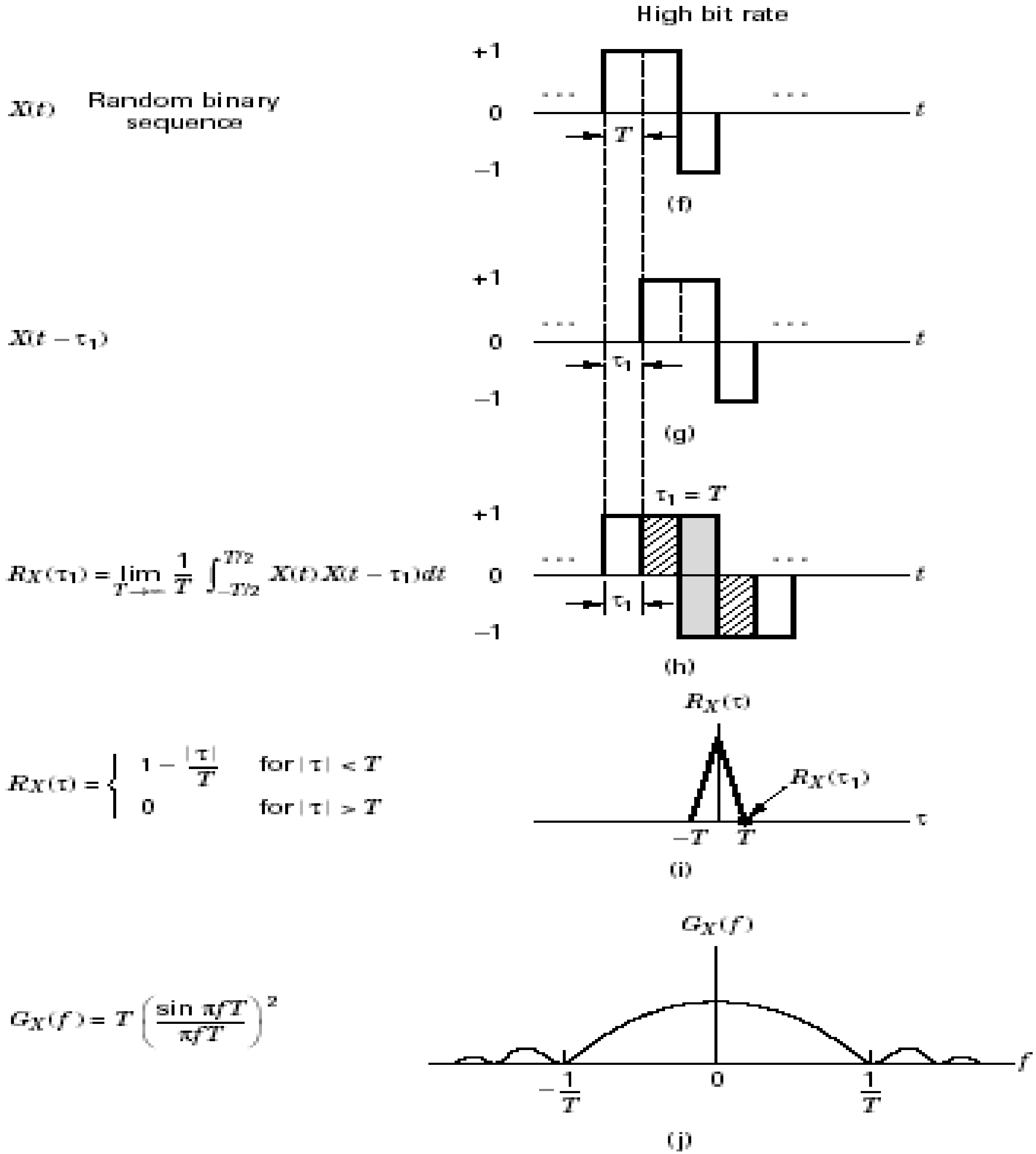


Figure 1.6 continued

Spectral Density

SPECTRAL DENSITY

- The *spectral density* of a signal characterizes the distribution of the signal’s energy or power, in the frequency domain
- This concept is particularly important when considering filtering in communication systems while evaluating the signal and noise at the filter output.
- The energy spectral density (ESD) or the power spectral density (PSD) is used in the evaluation.
- Need to determine how the average power or energy of the process is distributed in frequency.