Power and Energy Spectral Density

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- The power spectral density (PSD) $S_x(\omega)$ for a signal is a measure of its power distribution as a function of frequency
- It is a useful concept which allows us to determine the bandwidth required of a transmission system
- We will now present some basic results which will be employed later on

PSD

- Consider a signal *x(t)* with Fourier Transform (FT) *X(ω)*

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- We wish to find the energy and power distribution of *x(t)* as a function of frequency

Deterministic Signals

- If *x(t)* is the voltage across a *R*=1Ω resistor, the instantaneous power is,

$$\frac{(x(t))^2}{R} = (x(t))^2$$

- Thus the total energy in *x(t)* is,

$$\text{Energy} = \int_{-\infty}^{\infty} x(t)^2\, dt$$

- From Parseval's Theorem,

$$\text{Energy} = \int_{-\infty}^{\infty} \left|X(\omega)\right|^2 df$$

Deterministic Signals

- So,

$$\text{Energy} = \int_{-\infty}^{\infty} |X(\omega)|^2 df$$

$$= \int_{-\infty}^{\infty} |X(2\pi f)|^2 df$$

$$= \int_{-\infty}^{\infty} E(2\pi f) df$$

Where $E(2\pi f)$ is termed the Energy Density Spectrum (EDS), since the energy of $x(t)$ in the range f_o to $f_o + \delta f_o$ is,

$$E(2\pi f_o)\delta f_o$$

Deterministic Signals

- For communications signals, the energy is effectively infinite (the signals are of unlimited duration), so we usually work with *Power* quantities
- We find the average power by averaging over time

$$\text{Average power} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (x_T(t))^2 \, dt$$

Where $x_T(t)$ is the same as $x(t)$, but truncated to zero outside the time window $-T/2$ to $T/2$

- Using Parseval as before we obtain,

Deterministic Signals

$$\text{Average power} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (x_T(t))^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} \left| X_T(2\pi f) \right|^2 df$$

$$= \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{\left| X_T(2\pi f) \right|^2}{T} df$$

$$= \int_{-\infty}^{\infty} S_x(2\pi f) \, df$$

Where $S_x(\omega)$ is the Power Spectral Density (PSD)

PSD

$$S_x(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{T}$$

The power dissipated in the range f_o to $f_o + \delta f_o$ is,

$$S_x(2\pi f_o)\delta f_o$$

And $S_x(.)$ has units Watts/Hz

Wiener-Khintchine Theorem

- It can be shown that the PSD is also given by the FT of the autocorrelation function (ACF), $r_{xx}(\tau)$,

$$S_x(\omega) = \int_{-\infty}^{\infty} r_{xx}(\tau) e^{-j\omega\tau} d\tau$$

Where,

$$r_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

Random Signals

- The previous results apply to deterministic signals
- In general, we deal with random signals, eg the transmitted PAM signal is random because the symbols (a_k) take values at random
- Fortunately, our earlier results can be extended to cover random signals by the inclusion of an extra averaging or *expectation* step, over all possible values of the random signal $x(t)$

PSD, random signals

$$S_x(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{T}$$

Where $E[.]$ is the expectation operator

- The W-K result holds for random signals, choosing for *x(t)* any randomly selected realisation of the signal

Note: Only applies for ergodic signals where the time averages are the same as the corresponding ensemble averages

Linear Systems and Power Spectra

- Passing $x_T(t)$ through a linear filter $H(\omega)$ gives the output spectrum,

$$Y_T(\omega) = H(\omega) X_T(\omega)$$

- Hence, the output PSD is,

$$S_y(\omega) = \lim_{T \to \infty} \frac{E[|Y_T(\omega)|^2]}{T} = \lim_{T \to \infty} \frac{E[|H(\omega) X_T(\omega)|^2]}{T}$$

$$S_y(\omega) = |H(\omega)|^2 \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{T}$$

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$