Review of Probability and Random Processes

Importance of Random Processes

- Random variables and processes talk about quantities and signals which are unknown in advance
- The data sent through a communication system is modeled as random variable
- The noise, interference, and fading introduced by the channel can all be modeled as random processes
- Even the measure of performance (Probability of Bit Error) is expressed in terms of a probability

Random Events

- When we conduct a random experiment, we can use set notation to describe possible outcomes
- Examples: Roll a six-sided die

  Possible Outcomes: $S = \{1, 2, 3, 4, 5, 6\}$
- An *event* is any subset of possible outcomes: $A = \{1, 2\}$

Random Events (continued)

- The *complementary event*: $\overline{A} = S - A = \{3, 4, 5, 6\}$
- The set of all outcomes in the *certain event*: S
- The *null event*: ϕ
- Transmitting a data bit is also an experiment

Probability

- The probability P(A) is a number which measures the likelihood of the event A

Axioms of Probability

- No event has probability less than zero: $P(A) \geq 0$

  $P(A) \leq 1$ and $P(A) = 1 \Leftrightarrow A = S$
- Let A and B be two events such that: $A \cap B = \phi$

  Then: $P(A \cup B) = P(A) + P(B)$
- All other laws of probability follow from these axioms

Relationships Between Random Events

- Joint Probability: $P(AB) = P(A \cap B)$

  - Probability that both A and B occur

- Conditional Probability: $P(A \mid B) = \dfrac{P(AB)}{P(B)}$

  - Probability that A will occur given that B has occurred

Relationships Between Random Events

- Statistical Independence:

  - Events A and B are statistically independent if:

$$P(AB) = P(A)P(B)$$

  - If A and B are independence than:

$$P(A \mid B) = P(A) \quad \text{and} \quad P(B \mid A) = P(B)$$

Random Variables

- A random variable *X(S)* is a real valued function of the underlying even space: $s \in S$
- A random variable may be:

  -Discrete valued: range is *finite* (e.g.$\{0,1\}$) or *countable infinite* (e.g.$\{1,2,3\ldots..\}$)

  -Continuous valued: range is *uncountable infinite* (e.g. $\Re$)
- A random variable may be described by:

  - A name: X
  - Its range: $X \in \Re$
  - A description of its distribution

Cumulative Distribution Function

- Definition: $F_X(x) = F(x) = P(X \le x)$
- Properties:

  $\rightarrow F_X(x)$ is monotonically nondecreasing

  $\rightarrow F(-\infty) = 0$

  $\rightarrow F(\infty) = 1$

  $\rightarrow P(a < X \le b) = F(b) - F(a)$
- While the CDF defines the distribution of a random variable, we will usually work with the pdf or pmf
- In some texts, the CDF is called PDF (Probability Distribution function)

Probability Density Function

- Definition: $P_X(x) = \dfrac{dF_X(x)}{dx}$ or $P(x) = \dfrac{dF(x)}{dx}$
- Interpretations: *pdf* measures how fast the CDF is increasing or how likely a random variable is to lie around a particular value
- Properties:

$$P(x) \geq 0 \qquad \int_{-\infty}^{\infty} P(x)dx = 1$$

$$P(a < X \leq b) = \int_a^b P(x)dx$$

Expected Values

- Expected values are a shorthand way of describing a random variable
- The most important examples are:

-Mean: $$E(X) = m_x = \int_{-\infty}^{\infty} xp(x)dx$$

-Variance: $$E([X - m_x]^2) = \int_{-\infty}^{\infty} (x - m_x)^2 p(x)dx$$

Probability Mass Functions (pmf)

- A discrete random variable can be described by a pdf if we allow impulse functions
- We usually use probability mass functions (pmf)

$$p(x) = P(X = x)$$

- Properties are analogous to pdf

$$p(x) \geq 0$$

$$\sum_{X} p(x) = 1$$

$$P(a \leq X \leq b) = \sum_{x=a}^{b} p(x)$$

Some Useful Probability Distributions

- **Binary Distribution**

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$$

- This is frequently used for binary data
- Mean: $E(X) = p$
- Variance: $\sigma_X{}^2 = p(1-p)$

Some Useful Probability Distributions (continued)

- Let $Y = \sum_{i=1}^{n} X_i$ where $\{X_i, i = 1, \ldots, n\}$ are independent

  binary random variables with

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$$

- Then $p_Y(y) = \binom{n}{y} p^y (1-p)^{n-y} \qquad y = 0, 1, \ldots, n$
- Mean: $E(X) = np$
- Variance: $\sigma_X{}^2 = np(1-p)$

Some Useful Probability Distributions (continued)

- **Uniform pdf:**

$$p(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$$

- It is a continuous random variable

- Mean: $E(X) = \dfrac{1}{2}(a+b)$

- Variance: ${\sigma_X}^2 = \dfrac{1}{12}(a-b)^2$

Some Useful Probability Distributions (continued)

- **Gaussian pdf:** $p(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-(x-m_x)/2\sigma^2}$

- A gaussian random variable is completely determined by its mean and variance

The Q-function

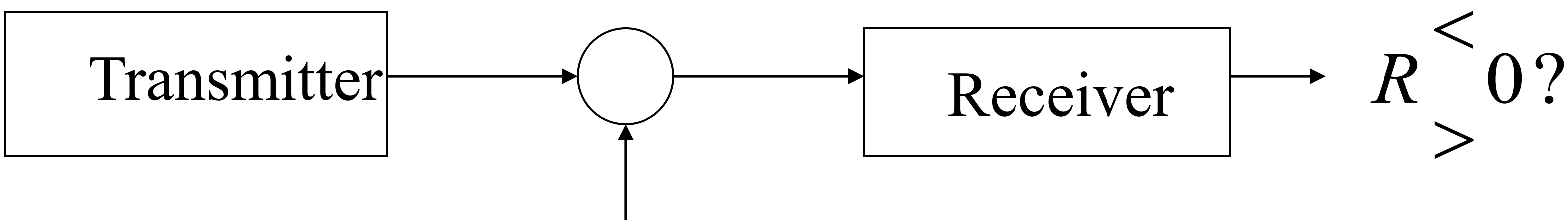
- The function that is frequently used for the area under the tail of the gaussian pdf is the denoted by Q(x)

$$Q(x) = \int_{x}^{\infty} e^{-t^2/2} dt, \qquad x \geq 0$$

- The Q-function is a standard form for expressing error probabilities without a closed form

A Communication System with Guassian noise

$$S \in \{\pm a\} \qquad R = S + N$$



N

- The probability that the receiver will make an error is

$$P(R>0 \mid S=-a) = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_n} e^{\frac{-(x+a)^2}{2\sigma_n^2}} dx = Q\left(\frac{a}{\sigma_n}\right)$$

Random Processes

- A random variable has a single value. However, actual signals change with time
- Random *variables* model unknown events
- Random *processes* model unknown signals
- A random process is just a collection of random variables
- If $X(t)$ is a random process, then $X(1)$, $X(1.5)$ and $X(37.5)$ are all random variables for any specific time t

Terminology Describing Random Processes

- A *stationary* random process has statistical properties which do not change at all time
- A *wide sense stationary* (WSS) process has a mean and autocorrelation function which do not change with time
- A random process is *ergodic* if the time average always converges to the statistical average
- Unless specified, we will assume that all random processes are WSS and ergodic

Description of Random Processes

- Knowing the pdf of individual samples of the random process is not sufficient.
  - We also need to know how individual samples are related to each other
- Two tools are available to decribe this relationship
  - Autocorrelation function
  - Power spectral density function

Autocorrelation

- Autocorrelation measures how a random process changes with time
- Intuitively, X(1) and X(1.1) will be strongly related than X(1) and X(100000)
- The autocorrelation function quantifies this
- For a WSS random process,

$$\phi_X(\tau) = E\left[X(t)X(t+\tau)\right]$$

- Note that $Power = \phi_X(0)$

Power Spectral Density

- $\Phi\left(f\right)$ tells us how much power is at each frequency
- Wiener-Khinchine Theorem: $\Phi(f) = F\{\phi(\tau)\}$
  - Power spectral density and autocorrelation are a Fourier Transform pair
- Properties of Power Spectral Density

$$\rightarrow \Phi(f) \geq 0$$

$$\rightarrow \Phi(f) = \Phi(-f)$$

$$\rightarrow Power = \int_{-\infty}^{\infty} \Phi(f) df$$

Gaussian Random Processes

- Gaussian random processes have some special properties
  - If a gaussian random process is wide-sense stationary, then it is also stationary
  - If the input to a linear system is a Gaussian random process, then the output is also a Gaussian random process

Linear systems

- Input: $x(t)$
- Impulse Response: $h(t)$
- Output: $y(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Computing the Output of Linear Systems

- Deterministic Signals:
  - Time domain: $y(t) = h(t) * x(t)$
  - Frequency domain: $Y(f) = F\{y(t)\} = X(f)H(f)$
- For a random process, we can still relate the statistical properties of the input and output signal
  - Time domain: $\phi_Y(\tau) = \phi_X(\tau) * h(\tau) * h(-\tau)$
  - Frequency domain: $\Phi_Y(f) = \Phi_X(f)\,|\,H(f)\,|^2$