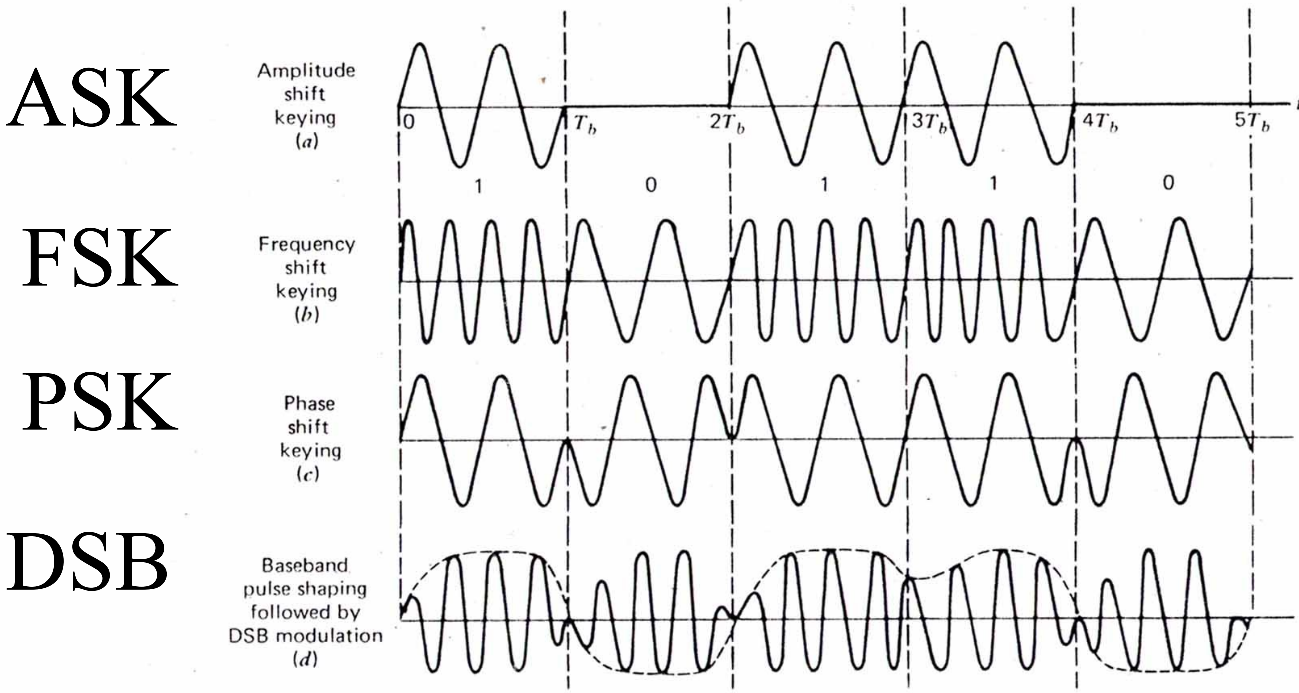
# General Expression for Error Probability of optimum receivers,

# INTRODUCTION

In order to transmit digital information over * bandpass channels, we have to transfer the information to a carrier wave of .appropriate frequency

We will study some of the most commonly * used digital modulation techniques wherein the digital information modifies the amplitude the phase, or the frequency of the carrier in .discrete steps

# The modulation waveforms for transmitting binary information over bandpass channels



# OPTIMUM RECEIVER FOR BINARY :DIGITAL MODULATION SCHEMS

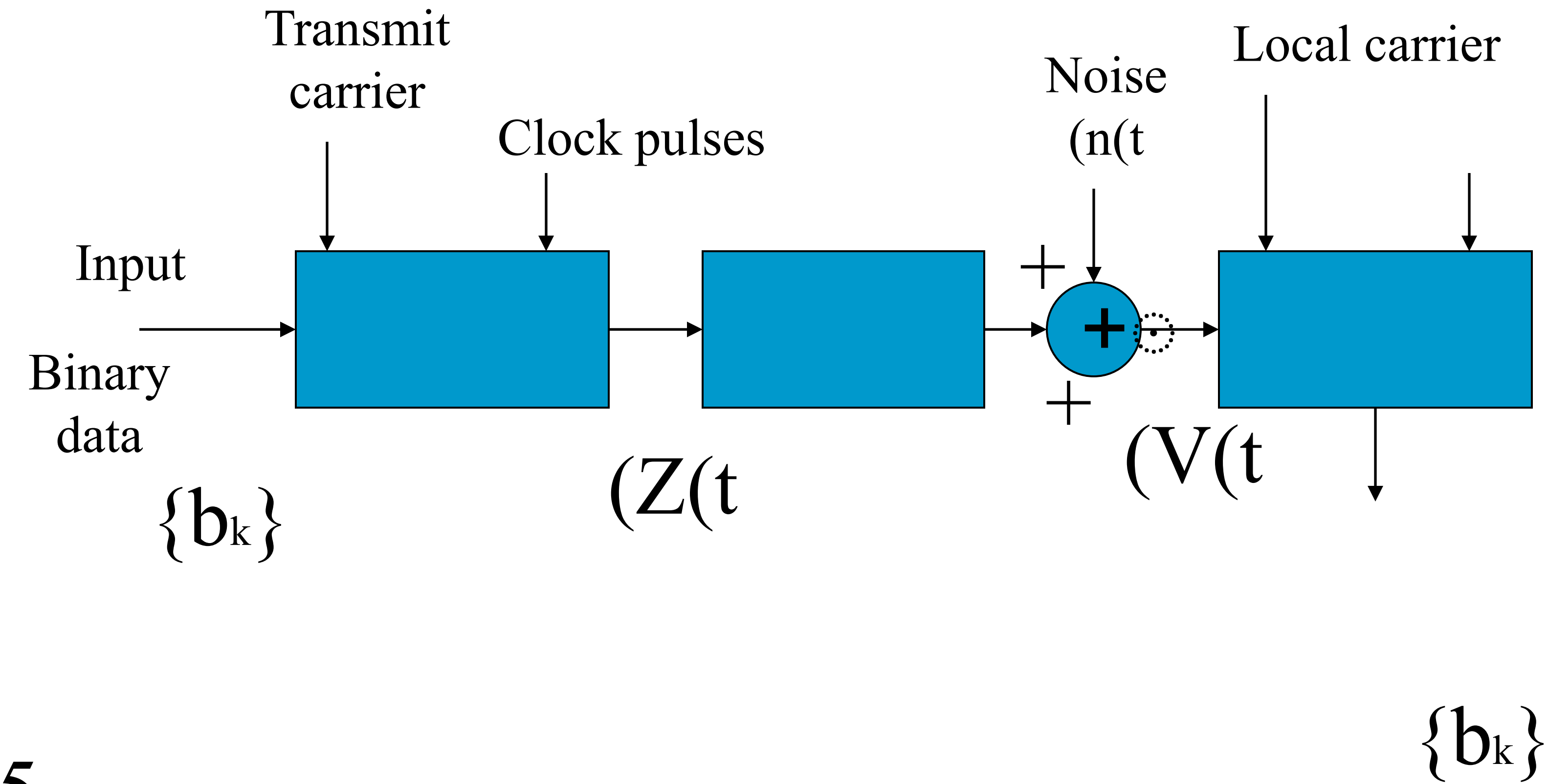
The function of a receiver in a binary communication * system is to distinguish between two transmitted signals .$S_1(t)$ and $S_2(t)$ in the presence of noise

The performance of the receiver is usually measured * in terms of the probability of error and the receiver is said to be optimum if it yields the minimum .probability of error

In this section, we will derive the structure of an optimum * receiver that can be used for demodulating binary .ASK,PSK,and FSK signals

# Description of binary ASK,PSK, and : FSK schemes

**-Bandpass binary data transmission system**



# :Explanation *

The input of the system is a binary bit sequence $\{b_k\}$ with a *
.bit rate $r_b$ and bit duration $T_b$

The output of the modulator during the Kth bit interval *
.depends on the Kth input bit $b_k$

The modulator output Z(t) during the Kth bit interval is *
a shifted version of one of two basic waveforms $S_1(t)$ or $S_2(t)$ and
:Z(t) is a random process defined by

.1

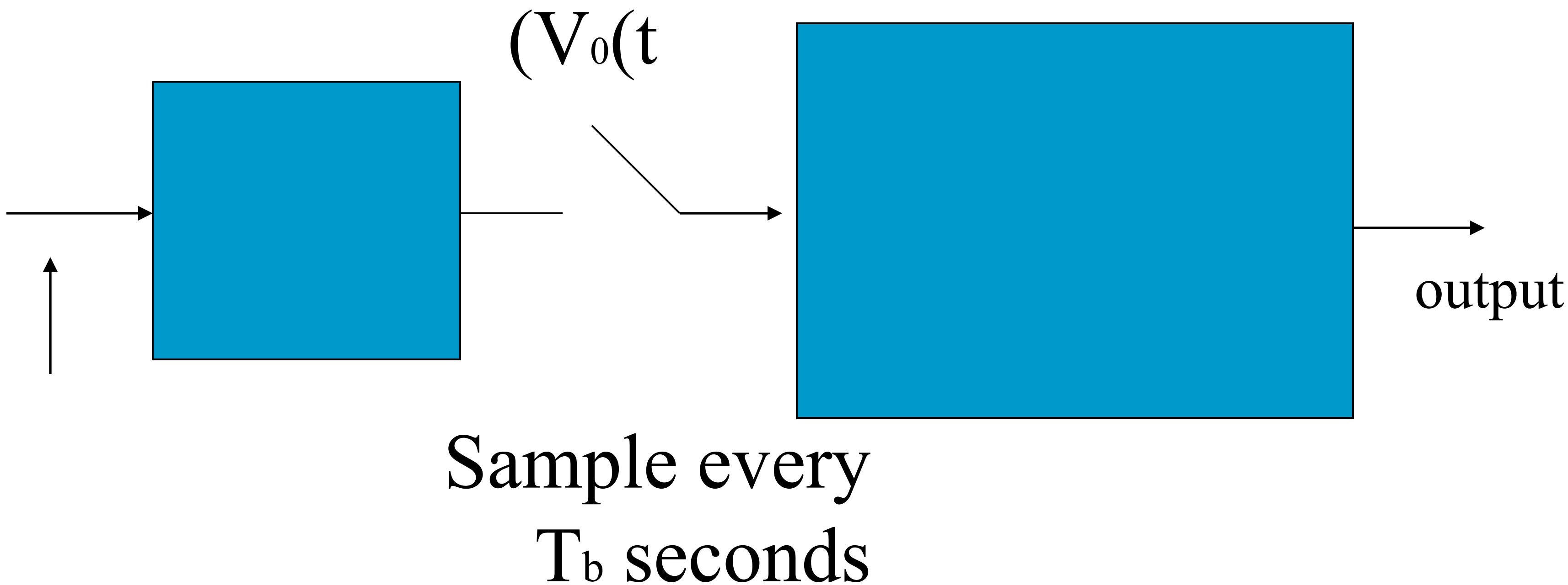
## Choice of signaling waveforms for various types of digital* modulation schemes

$S_1(t), S_2(t) = 0$ for $\rightarrow t \notin [0, T_b]; f_c = \frac{\omega_c}{2\pi}$

.The frequency of the carrier $f_c$ is assumed to be a multiple of $r_b$

| $S_1(T); 0 \le t \le T_b$ | $s_2(t); 0 \le t \le T_b$ | Type of modulation |
|---|---|---|
| $0$ | $A \cos w_c t$ (or $A \sin w_c t$) | ASK |
| $-A \cos w_c t$ $(or - A \sin w_c t)$ | $A \cos w_c t$ $(A \sin w_c t)$ | PSK |
| $A \cos\{(w_c - w_d)t\}$ $[(or\ A \sin\{(w_c - w_d)t\}]$ | $A \cos\{(w_c + w_d)t\}$ [or $A \sin\{(w_c + w_d)t\}$] | FSK |

# :Receiver structure



# :{Probability of Error-{$P_e$*

The measure of performance used for comparing *
!!!digital modulation schemes is the **probability of error**

The receiver makes errors in the decoding process *
!!! due to the noise present at its input

The receiver parameters as H(f) and threshold setting are *
!!!chosen to **minimize the probability of error**

## :The signal component in the output at t=kT$_b$

$$s_0(kT_b) = \int_{-\infty}^{kT_b} Z(\psi)h(kT_b - \psi)d\psi$$

$$= \int_{k-1)T_b}^{kT_b} Z(\psi)h(kT_b - \psi)d\psi + \text{ISI terms}$$

**h( $\psi$ ) is the impulse response of the receiver filter***

**ISI=0***

$$s_0(kT_b) = \int_{(k-1)T_b}^{kT_b} Z(\psi)h(kT_b - \psi)d\psi$$

**:The noise component $n_0(kT_b)$ is given by ***

The output noise $n_0(t)$ is a stationary zero mean Gaussian random process

**:The variance of $n_0(t)$ is***

**:The probability density function of $n_0(t)$ is***

# The probability that the kth bit is incorrectly decoded* :is given by

.2

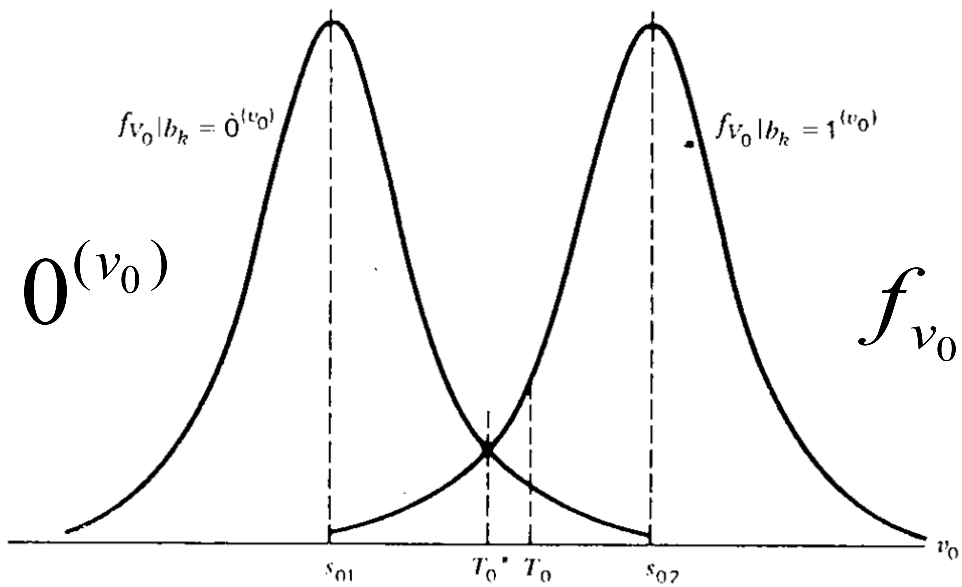
**:The conditional pdf of $V_0$ given $b_k = 0$ is given by***

.3

**:It is similarly when $b_k$ is 1***

# :Conditional pdf of $V_0$ given $b_k$

$$f_{v_0} \Big| b_k = 0^{(v_0)}$$




$$f_{v_0} \Big| \mathrm{b_k} = 1^{(v_0)}$$

## :The optimum value of the threshold $T_0$* is*

$$T_0^{\,*} = \frac{S_{01} + S_{02}}{2}$$

**Substituting the value of T*₀ for T₀ in equation 4* we can rewrite the expression for the probability :of error as**

$$P_e = \int_{(s_{02}-s_{01})/2}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{(V_0 - s_{01})^2}{2N_0}\right) dV_0$$

$$= \int_{(s_{02}-s_{01})/2\sqrt{N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right) dZ$$

The optimum filter is $\xi$ the filter that maximizes*
the ratio or the square of the ratio $\xi^2$
(maximizing $\xi^2$ eliminates the requirement $S_{01}<S_{02}$)

$$\xi = \frac{S_{02}(T_b) - S_{01}(T_b)}{\sqrt{N_0}}$$

## :Transfer Function of the Optimum Filter*

The probability of error is minimized by an *
appropriate choice of h(t) which maximizes $\xi^2$

**Where** →

$$\xi^2 = \frac{[s_{02}(T_b) - s_{01}(T_b)]^2}{N_0}$$

$$s_{02}(T_b) - s_{01}(T_b) = \int_0^{T_b} [s_2(\xi) - s_1(\xi)] h(T_b - \xi) d\xi$$

**And** →

$$N_0 = \int_{-\infty}^{\infty} G_n(f) \left| H(f) \right|^2 df$$

If we let P(t) =$S_2$(t)-$S_1$(t), then the numerator of the*
:quantity to be maximized is

$$S_{02}(T_b) - S_{01}(T_b) = P_0(T_b)$$

$$= \int_0^{T_b} P(\xi)h(T_b - \xi)d\gamma = \int_{-\infty}^{\infty} P(\xi)h(T_b - \xi)d\xi$$

Since P(t)=0 for t<0 and h( )$\lambda$=0 for $\lambda$<0*
:the Fourier transform of $P_0$ is

$$P_0(f) = P(f)H(f)$$

$$P_0(T_b) = \int_{-\infty}^{\infty} P(f)H(f)\exp(j2\pi f T_b)df$$

:Hence $\gamma^2$ can be written as*

(*)

We can maximize $\gamma^2$ by applying Schwarz's*
:inequality which has the form

(**)

Applying Schwarz's inequality to Equation(**) with-

and

$$X_1(f) = H(f)\sqrt{G_n(f)}$$

$$X_2(f) = \frac{P(f)\exp(j2\pi f T_b)}{\sqrt{G_n(f)}}$$

We see that H(f), which maximizes $\zeta$ is given by-

(***)

**!!! Where K is an arbitrary constant**

Substituting equation (***) in(*) , we obtain-
:the maximum value of $\gamma^2$ as

:And the minimum probability of error is given by-

$$P_e = \int_{\gamma_{\max/2}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right) dZ = Q\left(\frac{\gamma_{\max}}{2}\right)$$

# :Matched Filter Receiver*

If the channel noise is white, that is, $G_n(f) = \eta/2$, then the transfer - :function of the optimum receiver is given by

From Equation (***) with the arbitrary constant K set equal to $\eta/2$ -

**:The impulse response of the optimum filter is**

Recognizing the fact that the inverse Fourier *
of $P^*(f)$ is $P(-t)$ and that $\exp(-2 \quad j\pi T_b)$ represent
:a delay of $T_b$ we obtain h(t) as

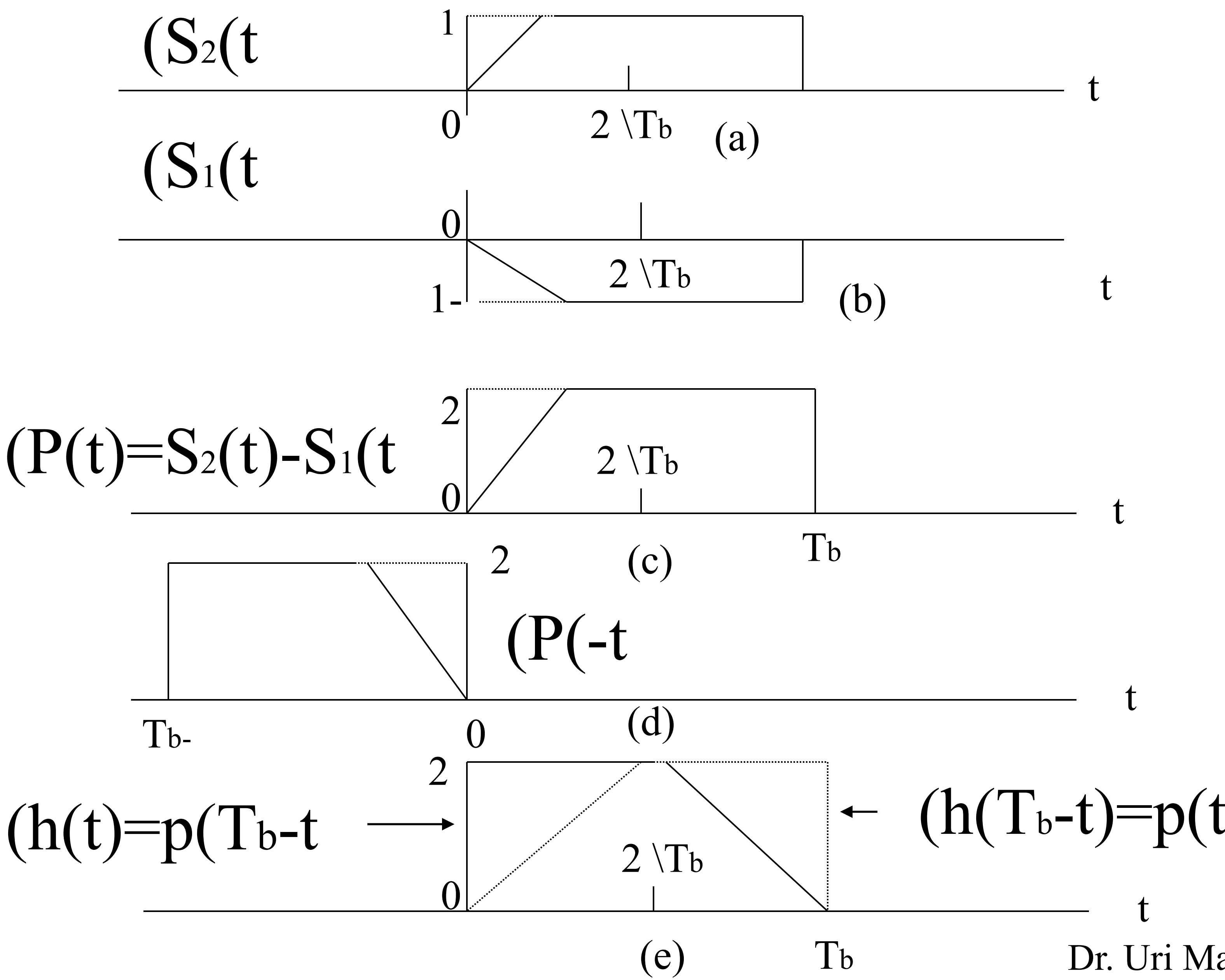
$$h(t) = p(T_b - t)$$

:Since $p(t)=S_1(t)-S_2(t)$ , we have*

The impulse response h(t) is matched to the signal *
:$S_1(t)$ and $S_2(t)$ and for this reason the filter is called

MATCHED FILTER

# :Impulse response of the Matched Filter *



# :Correlation Receiver*

The output of the receiver at t=Tb*

Where V($\xi$) is the noisy input to the receiver

Substituting $h(\xi) = S_2(T_b - \xi) - S_1(T_b - \xi)$ and noting *

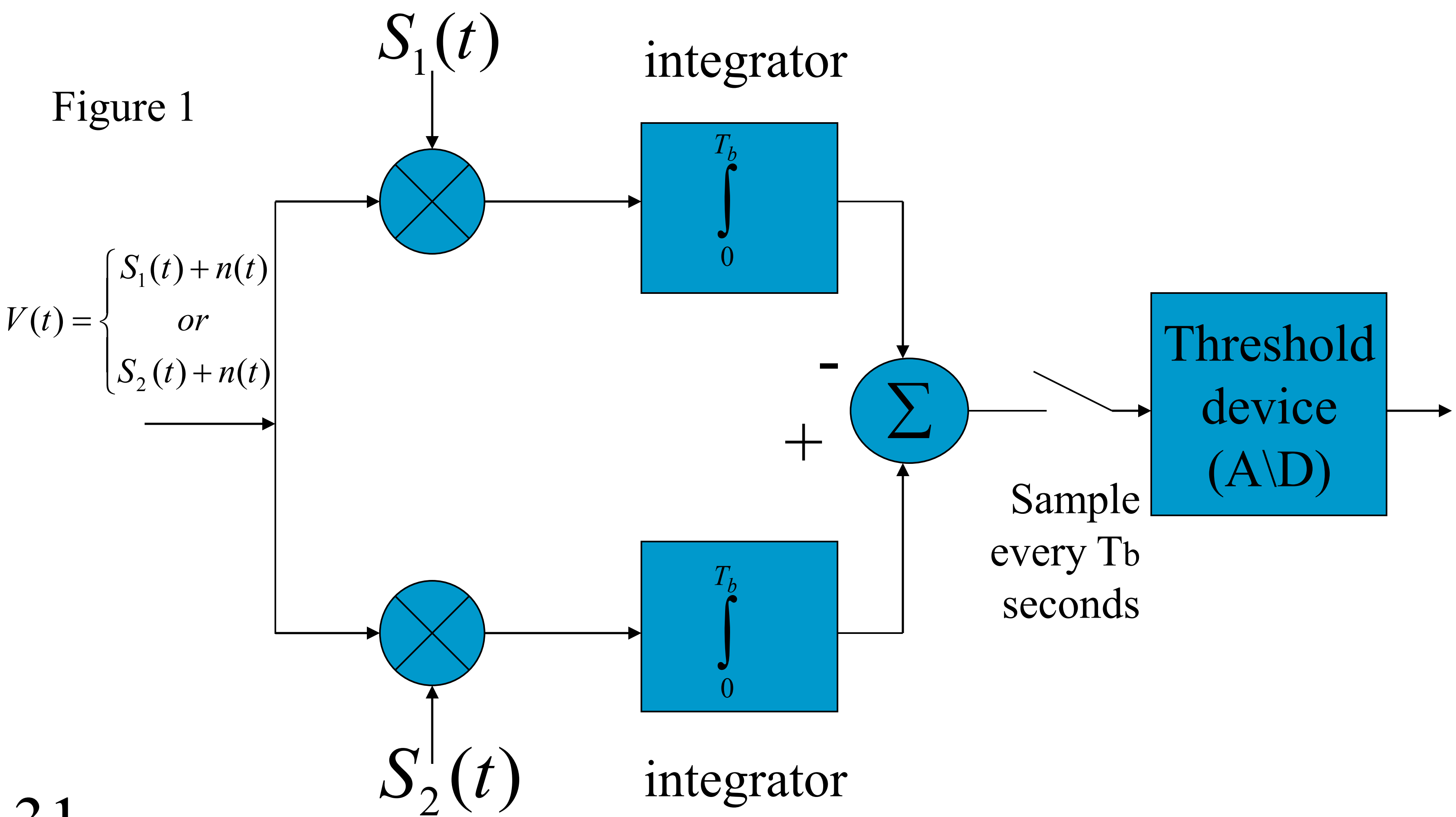
: that $h(\xi) = 0 \text{ for } \xi \notin (0, \mathrm{T_b})$ we can rewrite the preceding expression as

(# #)

Equation(# #) suggested that the optimum receiver can be implemented * as shown in Figure 1 .This form of the receiver is called

# A Correlation Receiver

Figure 1



In actual practice, the receiver shown in Figure 1 is actually * .implemented as shown in Figure 2

In this implementation, the integrator has to be reset at the - (end of each signaling interval in order to ovoid (I.S.I

**!!! Inter symbol interference**

## :Integrate and dump correlation receiver

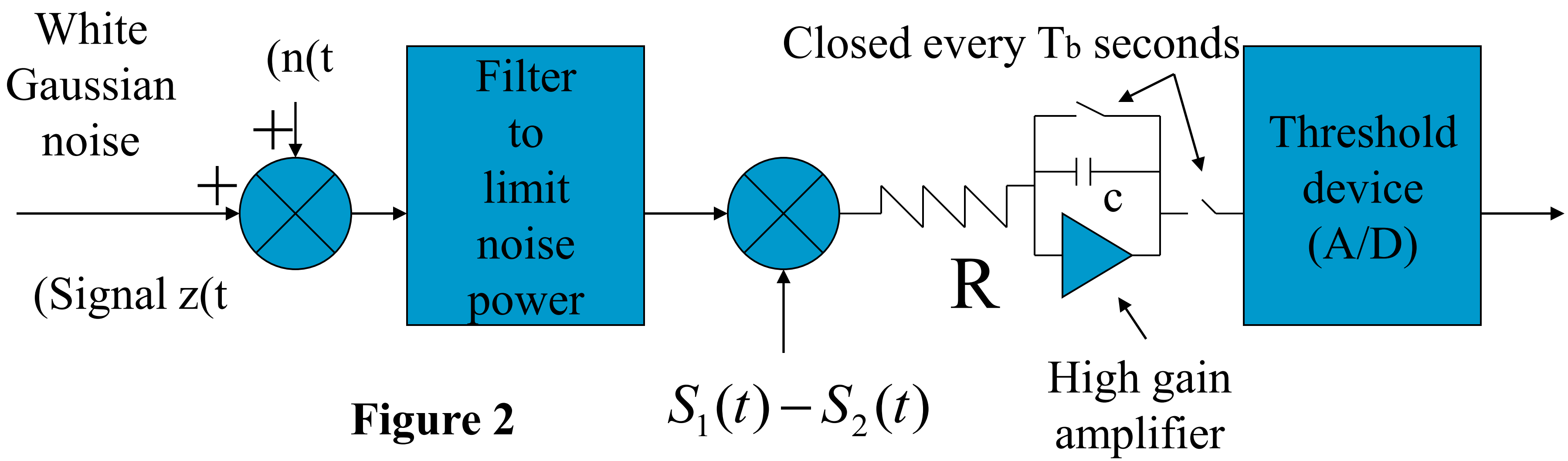



**Figure 2**

The bandwidth of the filter preceding the integrator is assumed * !!! to be wide enough to pass z(t) without distortion

**Example**: A band pass data transmission scheme uses a PSK signaling scheme with

$$S_2(t) = A\cos w_c t,\ 0 \le \mathrm{t} \le \mathrm{T_b}\ , w_c = 10\pi / T_b$$

$$S_1(t) = -A\cos w_c t,\ 0 \le \mathrm{t} \le \mathrm{T_b}\ , \mathrm{T_b} = 0.2m\sec$$

The carrier amplitude at the receiver input is 1 mvolt and the psd of the A.W.G.N at input is $10^{-11}$ watt/Hz. Assume that an ideal correlation receiver is used. Calculate the .average bit error rate of the receiver

# :Solution

Data rate =5000 bit/sec

$$G_n(f)=\eta/2=10^{-11}\ watt/Hz$$

Receiver impulse response $= h(t)=S_2(T_b-t)-S_1(T_b-t)$

$$=2A\cos w_c(T_b-t)$$

Threshold setting is 0 and

$$\gamma^2{}_{\max}=\int_{-\infty}^{\infty}\frac{|P(f)|^2}{G_n(f)}df$$

$$=\left(\frac{2}{\eta}\right)\int_{-\infty}^{\infty}|P(f)|^2\,dt$$

$$=\left(\frac{2}{\eta}\right)\int_{0}^{T_b}[S_2(t)-S_1(t)]^2\,dt$$

$$=\left(\frac{2}{\eta}\right)\int_{0}^{T_b}4A^2(\cos w_c t)^2\,dt$$

$$=\left(\frac{2}{\eta}\right)(2A^2T_b)=\frac{4A^2T_b}{\eta}=40$$

# :Solution Continue

**=Probability of error = Pe ***

$$= \int_{\frac{1}{2}\gamma_{max}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz$$

$$= Q\left(\sqrt{10}\right)$$

From the table of Gaussian probabilities ,we*
get $P_e \cong 0.0008$ and
Average error rate $\cong (r_b)\, p_e$/sec = 4 bits/sec