

ELEMENTARY FLUID DYNAMICS: THE BERNOULLI EQUATION



Bernoulli Along a Streamline

$$-\tilde{N}p = r \mathbf{a} + g \hat{\mathbf{k}} \quad (\text{eqn 2.2})$$

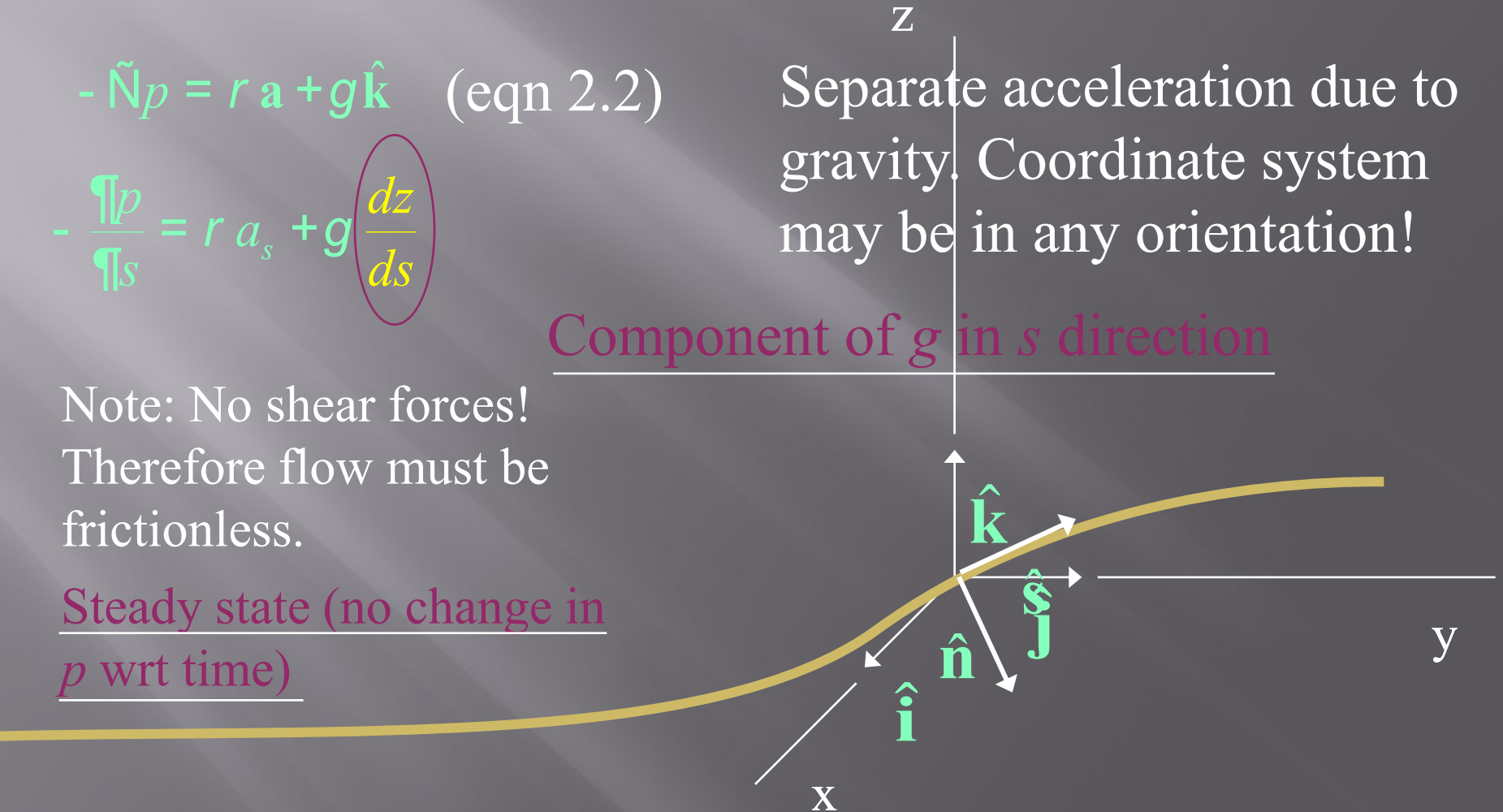
$$-\frac{\nabla p}{\rho} = r a_s + g \left(\frac{dz}{ds} \right)$$

Note: No shear forces!
Therefore flow must be frictionless.

Steady state (no change in p wrt time)

Separate acceleration due to gravity. Coordinate system may be in any orientation!

Component of g in s direction



Bernoulli Along a Streamline

$$-\frac{\nabla p}{\nabla s} = r a_s + g \frac{dz}{ds}$$

Can we eliminate the partial derivative?

chain rule

$$a_s = \frac{dV}{dt} = \frac{\nabla V}{\nabla s} \frac{ds}{dt} = \frac{\nabla V}{\nabla s} V$$

Write acceleration as derivative wrt s

$$dp = \frac{\nabla p}{\nabla s} ds + \frac{\nabla p}{\nabla n} dn \quad \begin{array}{l} \nearrow 0 \text{ (} n \text{ is constant along streamline)} \\ \searrow dp/ds = \nabla p / \nabla s \text{ and } dV/ds = \nabla V / \nabla s \end{array}$$

$$-\frac{dp}{ds} = r V \frac{dV}{ds} + g \frac{dz}{ds}$$

$$V \frac{dV}{ds} = \frac{1}{2} \frac{d(V^2)}{ds}$$

Integrate $F=ma$ Along a Streamline

$$-\frac{dp}{ds} = \frac{1}{2} r \frac{d(V^2)}{ds} + g \frac{dz}{ds}$$

$$dp + \frac{1}{2} r d(V^2) + g dz = 0$$

$$\circ \frac{dp}{r} + \frac{1}{2} \circ d(V^2) + g \circ dz = 0$$

$$\circ \frac{dp}{r} + \frac{1}{2} V^2 + gz = C$$

$$p + \frac{1}{2} r V^2 + gz = C$$

Eliminate ds

Now let's integrate...
But density is a function
of pressure.

If density is constant...

Along a streamline

Bernoulli Equation

- ▣ Assumptions needed for Bernoulli Equation
 - Inviscid (frictionless)
 - Steady
 - Constant density (incompressible)
 - Along a streamline

- ▣ Eliminate the constant in the Bernoulli equation?
Apply at two points along a streamline.

- ▣ Bernoulli equation does not include
 - Mechanical energy to thermal energy
 - Heat transfer, shaft work

Bernoulli Equation

The Bernoulli Equation is a statement of the conservation of Mechanical Energy

$$\frac{p}{\rho} + gz + \frac{1}{2}V^2 = C$$

↓ ↓
p.e. k.e.

$$\frac{p}{\rho g} + z + \frac{V^2}{2g} = C$$

$$\frac{p}{\rho g} = \text{Pressure head}$$

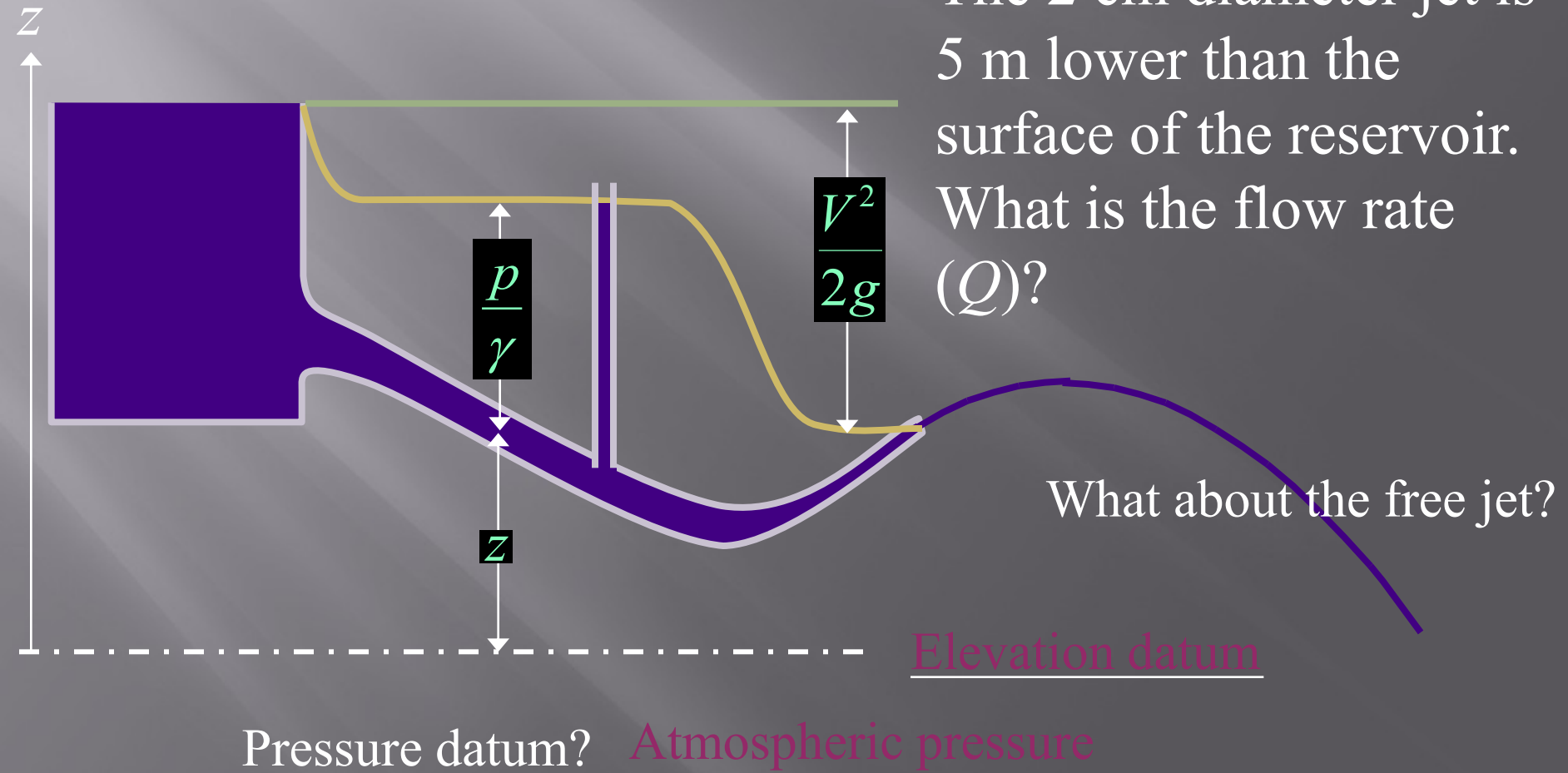
$$z = \text{Elevation head}$$

$$\frac{V^2}{2g} = \text{Velocity head}$$

$$\frac{p}{\rho g} + z = \text{Piezometric head}$$

$$\frac{p}{\rho g} + z + \frac{V^2}{2g} = \text{Total head}$$

Hydraulic and Energy Grade Lines (neglecting losses for now)



Bernoulli Equation: Simple Case ($V = 0$)

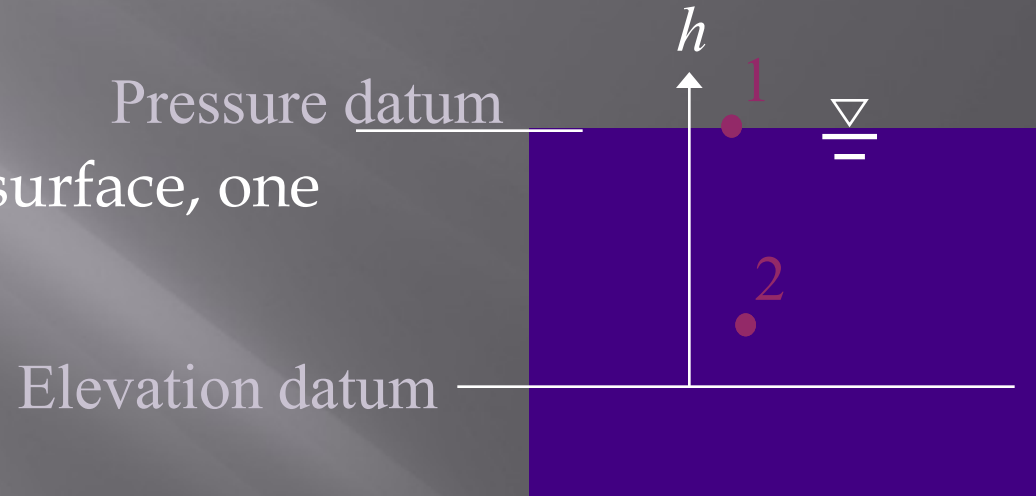
▣ Reservoir ($V = 0$)

- ▣ Put one point on the surface, one point anywhere else

$$\frac{p}{g} + z + \frac{V^2}{2g} = C$$

$$\frac{p_1}{g} + z_1 = \frac{p_2}{g} + z_2$$

$$z_1 - z_2 = \frac{p_2}{g}$$



We didn't cross any streamlines
so this analysis is okay!

Same as we found using statics

Bernoulli Equation: Simple Case ($p = 0$ or constant)

- What is an example of a fluid experiencing a change in elevation, but remaining at a constant pressure? _____

Free jet

$$\cancel{\frac{p_1}{g}} + z_1 + \frac{V_1^2}{2g} = \cancel{\frac{p_2}{g}} + z_2 + \frac{V_2^2}{2g}$$

$$z_1 + \frac{V_1^2}{2g} = z_2 + \frac{V_2^2}{2g}$$

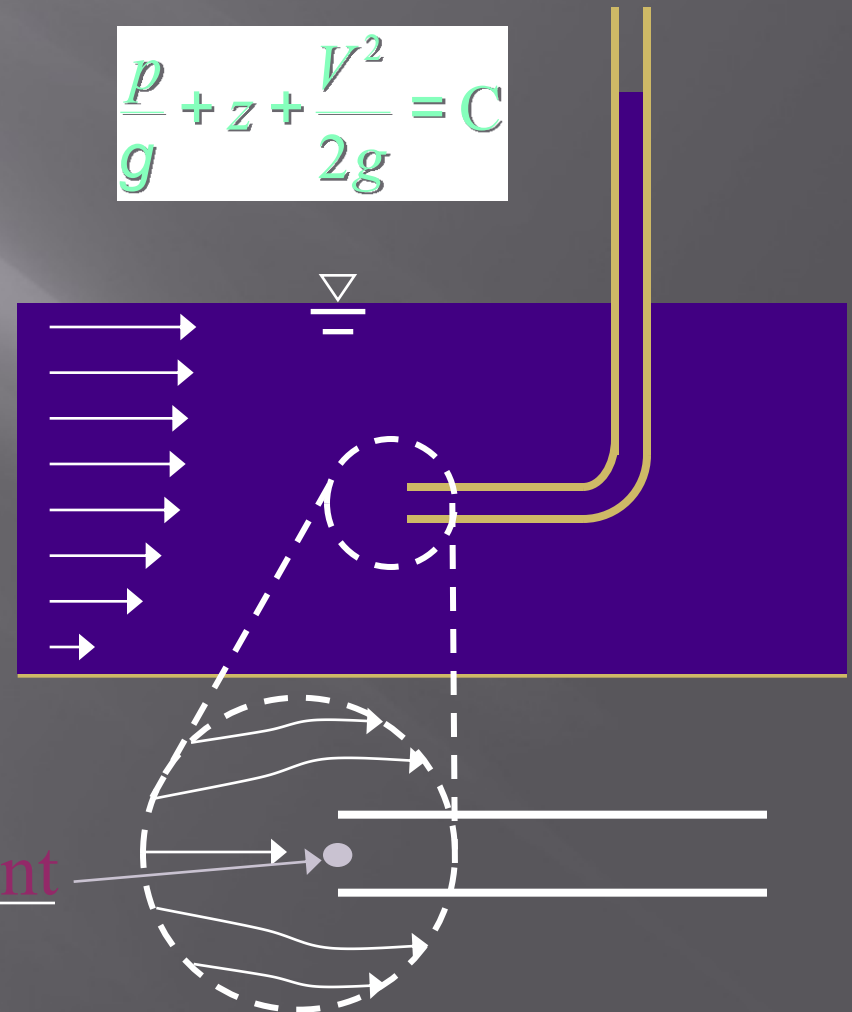
$$V_2 = \sqrt{2g(z_1 - z_2) + V_1^2}$$



Bernoulli Equation Application: Stagnation Tube

- What happens when the water starts flowing in the channel?
- Does the orientation of the tube matter? Yes!
- How high does the water rise in the stagnation tube?
- How do we choose the points on the streamline?

$$\frac{p}{\rho} + z + \frac{V^2}{2g} = C$$



Bernoulli Equation Application: Stagnation Tube

□ 1a-2a

▪ Same streamline

□ 1b-2a

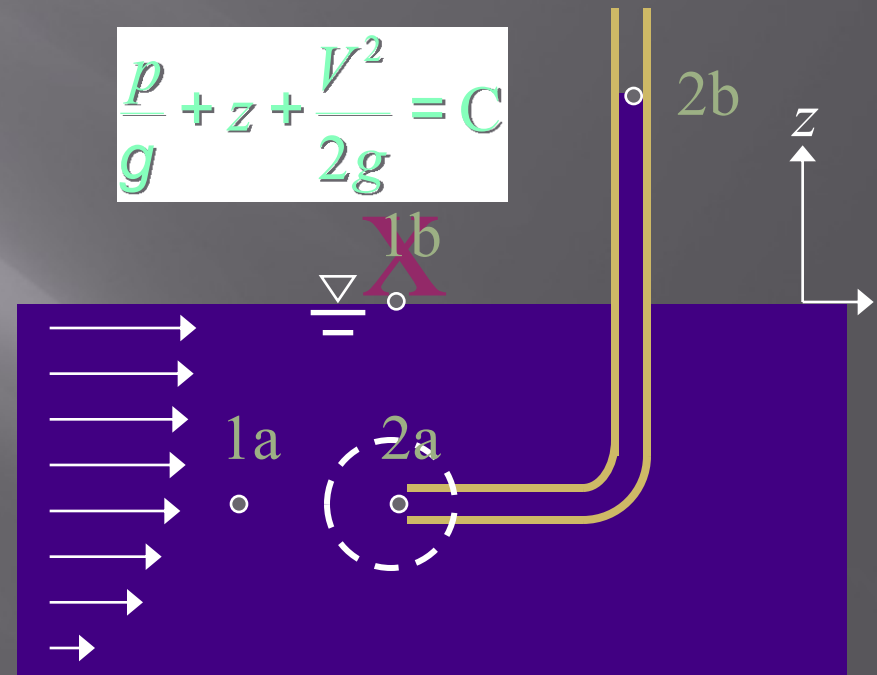
▪ Crosses streamlines

□ 1a-2b

▪ Doesn't cross streamlines

$$\frac{p_1}{g} + z_1 + \frac{V_1^2}{2g} = \cancel{\frac{p_2}{g}} + z_2 + \cancel{\frac{V_2^2}{2g}}$$

$$\frac{p}{g} + z + \frac{V^2}{2g} = C$$



1. We can obtain V_1 if p_1 and $(z_2 - z_1)$ are known
2. z_2 is the total energy!

Stagnation Tube

- ▣ Great for measuring _____
- ▣ How could you measure Q ?
- ▣ Could you use a stagnation tube in a pipeline?
 - What problem might you encounter?
 - **How could you modify the stagnation tube to solve the problem?**

Bernoulli

Normal to the Streamlines

$$-\tilde{N}p = r a + g \hat{k}$$

$$-\frac{\nabla p}{\nabla n} = r a_n + g \frac{dz}{dn}$$

Separate acceleration due to gravity. Coordinate system may be in any orientation!

Component of g in n direction



Bernoulli

Normal to the Streamlines

$$-\frac{\rho}{\rho} = r a_n + g \frac{dz}{dn}$$

$$a_n = \frac{V^2}{R}$$

centrifugal force. R is local radius of curvature
 n is toward the center of the radius of curvature

0 (s is constant along streamline)

$$dp = \frac{\rho}{\rho} ds + \frac{\rho}{\rho} dn \quad \setminus \quad dp/dn = \rho/\rho \quad \text{and} \quad dV/dn = \rho V/\rho$$

$$-\frac{dp}{dn} = r \frac{V^2}{R} + g \frac{dz}{dn}$$

Integrate $F=ma$ Normal to the Streamlines

$$-\frac{dp}{dn} = r \frac{V^2}{R} + g \frac{dz}{dn}$$

Multiply by dn

$$\int \frac{dp}{r} + \int \frac{V^2}{R} dn + \int g dz = C$$

Integrate

$$\frac{p}{r} + \int \frac{V^2}{R} dn + gz = C$$

If density is constant...

$$p + r \int \frac{V^2}{R} dn + gz = C$$

Normal to streamline

Pressure Change Across Streamlines

$$p + r \frac{V^2}{R} dn + g_z = C$$

If you cross streamlines that are straight and parallel, then $p + g_z = C$ and the pressure is hydrostatic.

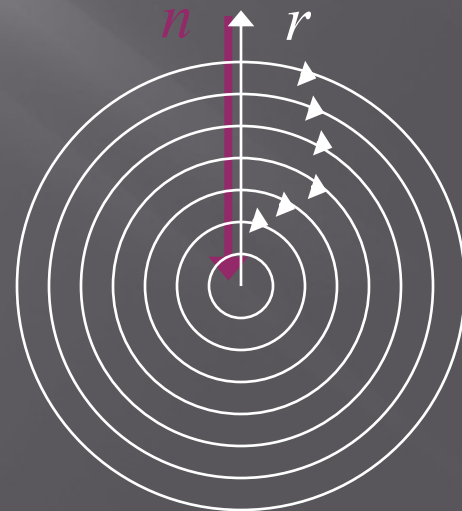
$$p - r C_1^2 \int r dr + g_z = C$$

$$V(r) = C_1 r$$

$$dn = -dr$$

$$p - \frac{r C_1^2}{2} r^2 + g_z = C$$

As r decreases p decreases



Pitot Tubes



- ▣ Used to measure air speed on airplanes
- ▣ Can connect a differential pressure transducer to directly measure $V^2/2g$
- ▣ Can be used to measure the flow of water in pipelines

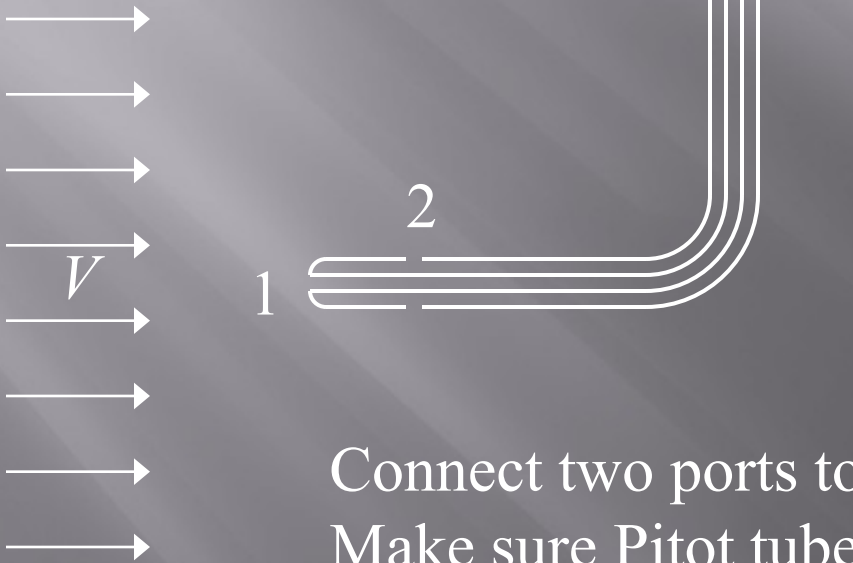
Point measurement!



Pitot Tube

Stagnation pressure tap

Static pressure tap



$$\frac{p_1}{g} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{g} + z_2 + \frac{V_2^2}{2g}$$

$$V_1 = \underline{0}$$

$$z_1 = z_2$$

$$V = \sqrt{\frac{2}{\rho} (p_1 - p_2)}$$

Connect two ports to differential pressure transducer.
Make sure Pitot tube is completely filled with the fluid that is being measured.

Solve for velocity as function of pressure difference

Relaxed Assumptions for Bernoulli Equation

- ▣ Frictionless
- ▣ Steady Viscous energy loss must be small
- ▣ Constant density (incompressible)
Or gradually varying
- ▣ Along a streamline
Small changes in density
- Don't cross streamlines

Bernoulli Equation Applications

- ▣ Stagnation tube
- ▣ Pitot tube
- ▣ Free Jets
- ▣ Orifice
- ▣ Venturi
- ▣ Sluice gate
- ▣ Sharp-crested weir

Applicable to **contracting** streamlines (accelerating flow).

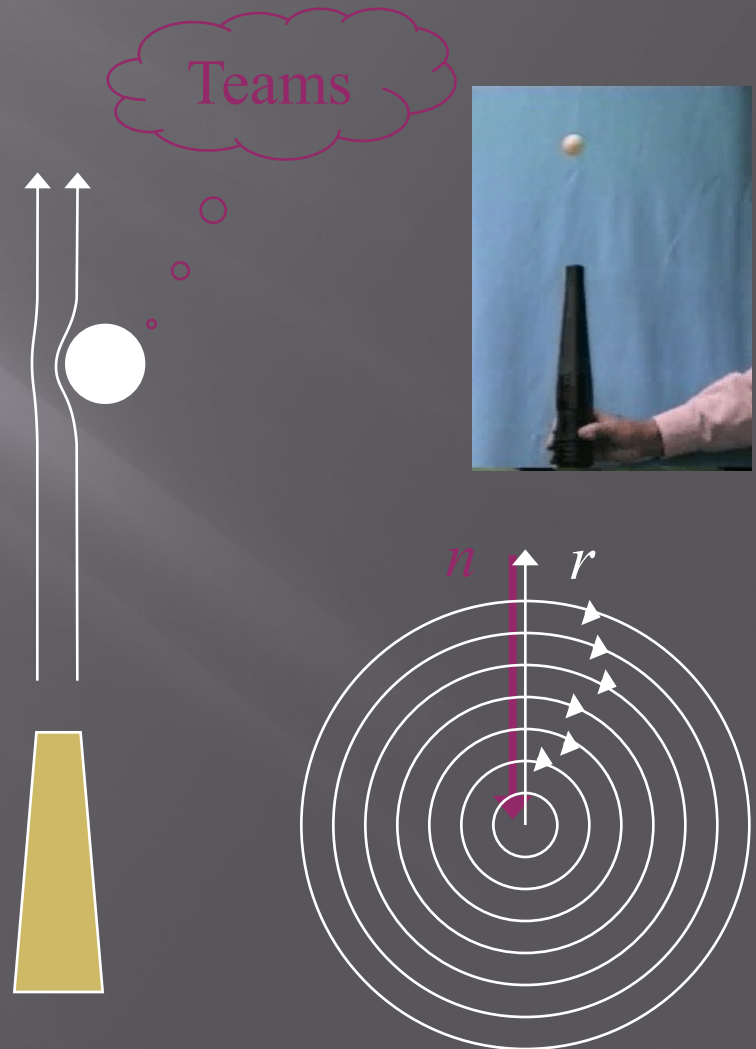


Ping Pong Ball

Why does the ping pong ball try to return to the center of the jet?

What forces are acting on the ball when it is not centered on the jet?

How does the ball choose the distance above the source of the jet?



Summary

- ▣ By integrating $F=ma$ along a streamline we found...
 - That energy can be converted between pressure, elevation, and velocity
 - That we can understand many simple flows by applying the Bernoulli equation
- ▣ However, the Bernoulli equation can not be applied to flows where viscosity is large or where mechanical energy is converted into thermal energy.

Jet Problem

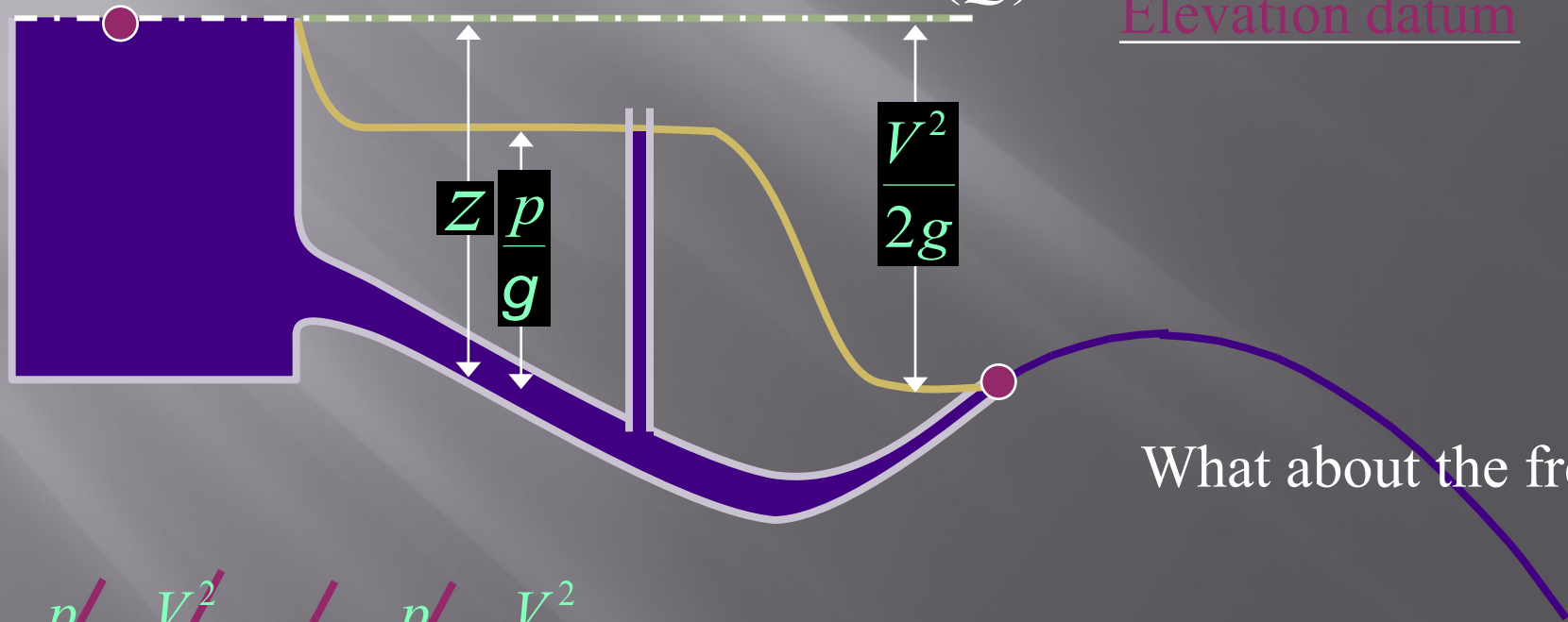
- ▣ How could you choose your elevation datum to help simplify the problem?
- ▣ How can you pick 2 locations where you know enough of the parameters to actually find the velocity?
- ▣ You have one equation (so one unknown!)



Jet Solution

The 2 cm diameter jet is 5 m lower than the surface of the reservoir. What is the flow rate (Q)?

Elevation datum

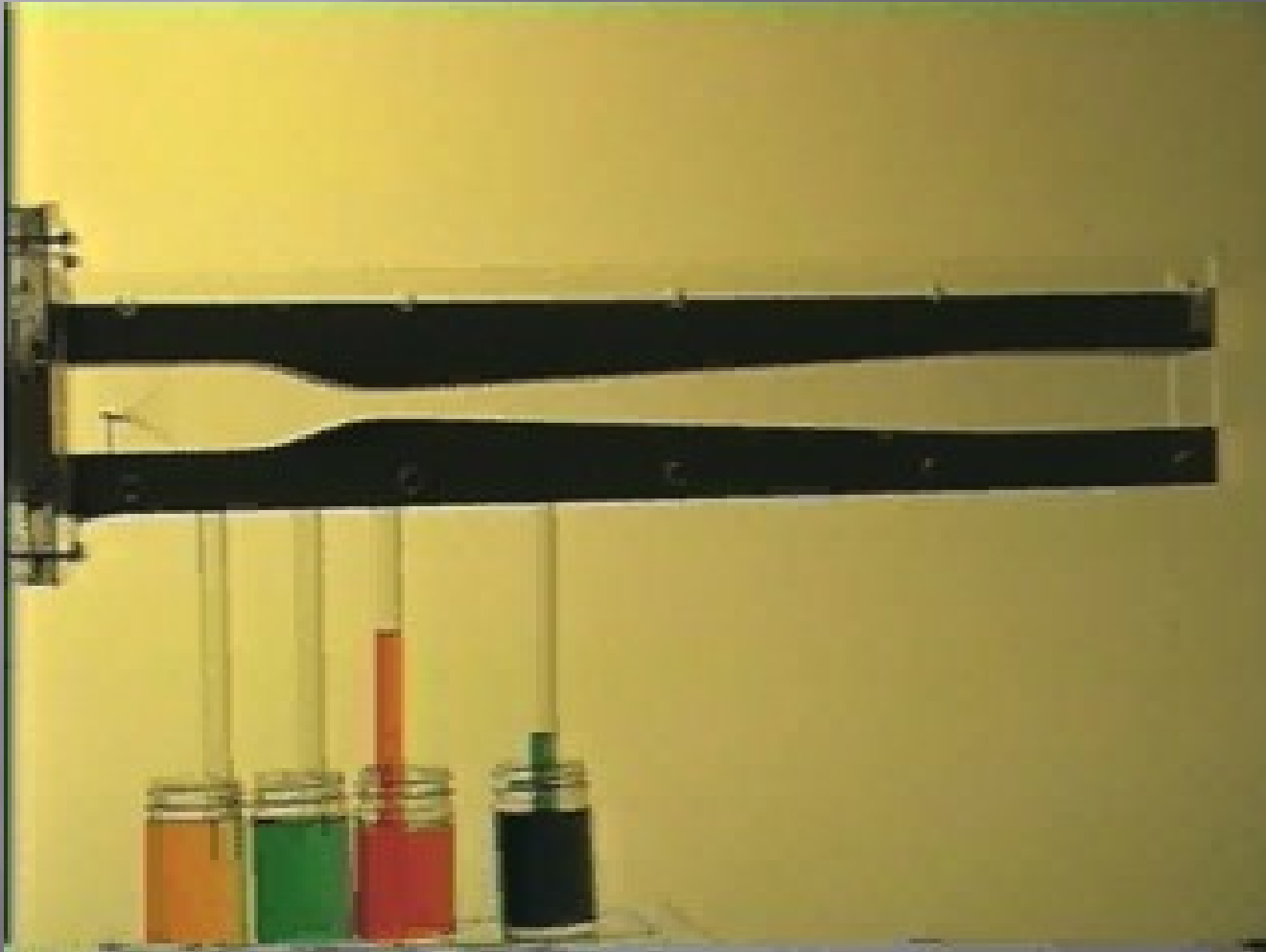


What about the free jet?

$$\cancel{\frac{p_1}{g}} + \cancel{\frac{V_1^2}{2g}} + \cancel{z_1} = \cancel{\frac{p_2}{g}} + \frac{V_2^2}{2g} + z_2 \quad z_2 = -5 \text{ m}$$

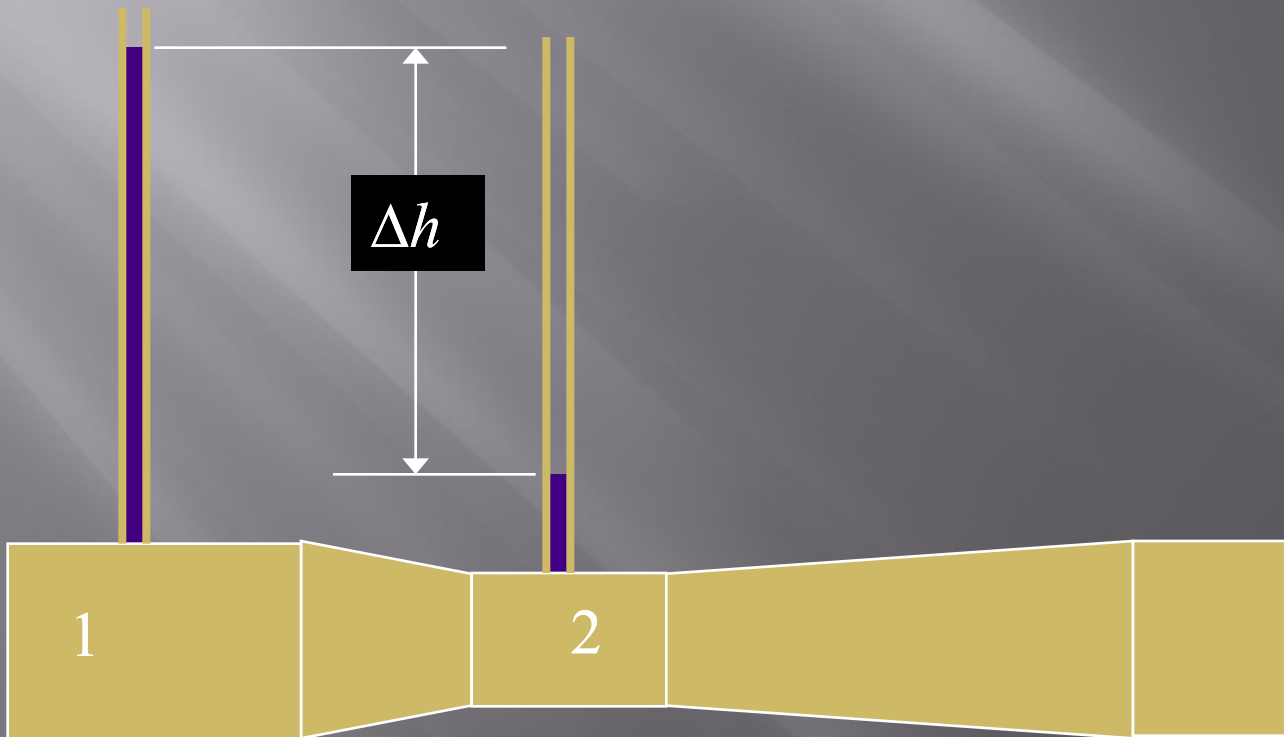


Example: Venturi



Example: Venturi

Find the flow (Q) given the pressure drop between point 1 and 2 and the diameters of the two sections. You may assume the head loss is negligible. Draw the EGL and the HGL.



Example Venturi

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g} \left[1 - \left(\frac{d_2}{d_1} \right)^4 \right]$$

$$V_2 = \sqrt{\frac{2g(p_1 - p_2)}{\gamma \left[1 - (d_2/d_1)^4 \right]}}$$

$$Q = C_v A_2 \sqrt{\frac{2g(p_1 - p_2)}{\gamma \left[1 - (d_2/d_1)^4 \right]}}$$

$$Q = VA$$

$$V_1 A_1 = V_2 A_2$$

$$V_1 \frac{\pi d_1^2}{4} = V_2 \frac{\pi d_2^2}{4}$$

$$V_1 d_1^2 = V_2 d_2^2$$

$$V_1 = V_2 \frac{d_2^2}{d_1^2}$$

