

* Cauchy's Integral Theorem and Formula

Cauchy's integral Theorem

We now turn to integration.

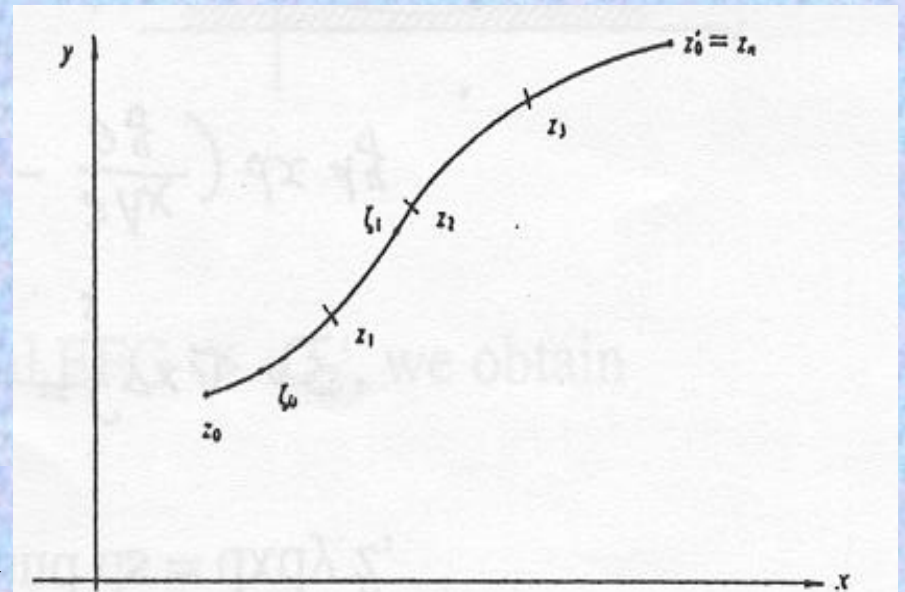
in close analogy to the integral of a real function

The contour $z_0 \rightarrow z_n$ is divided into n intervals. Let $n \rightarrow \infty$

with $|\Delta z_j| = |z_j - z_{j-1}| \rightarrow 0$ for j . Then

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n f(\zeta_j) \Delta z_j = \int_{z_0}^{z_n} f(z) dz$$

provided that the limit exists and is independent of the details of choosing the points z_j and ζ_j , where ζ_j is a point on the curve between z_j and z_{j-1} .



The right-hand side of the above equation is called the contour (path) integral of $f(z)$

As an alternative, the contour may be defined by

$$\begin{aligned}\int_C^{z_1}^{z_2} f(z) dz &= \int_C^{x_1, y_1}^{x_2, y_2} [u(x, y) + iv(x, y)] [dx + idy] \\ &= \int_C^{x_1, y_1}^{x_2, y_2} [u dx - v dy] + i \int_C^{x_1, y_1}^{x_2, y_2} [v dx + u dy]\end{aligned}$$

with the path C specified. This reduces the complex integral to the complex sum of real integrals. It's somewhat analogous to the case of the vector integral.

An important example $\int_C z^n dz$

where C is a circle of radius $r > 0$ around the origin $z=0$ in the direction of counterclockwise.

In polar coordinates, we parameterize
and $dz = ire^{i\theta} d\theta$ and have

$$z = re^{i\theta}$$

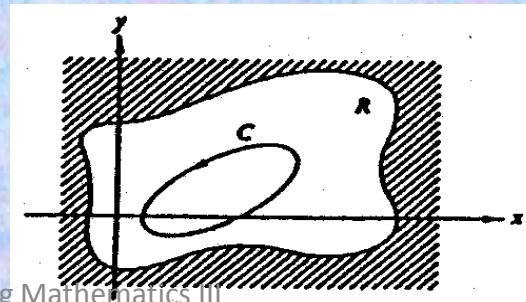
$$\frac{1}{2\pi i} \int_c z^n dz = \frac{r^{n+1}}{2\pi} \int_0^{2\pi} \exp[i(n+1)\theta] d\theta$$
$$= \begin{cases} 0 & \text{for } n \neq -1 \\ 1 & \text{for } n = -1 \end{cases}$$

which is independent of r.

Cauchy's integral theorem

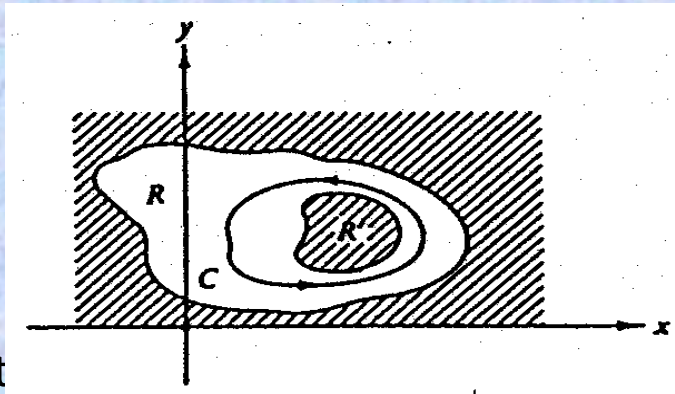
- If a function $f(z)$ is analytical (therefore single-valued) [and its partial derivatives are continuous] through some simply connected region R , for every closed path C in R ,

$$\oint_c f(z) dz = 0$$



• *Multiply connected regions*

The original statement of our theorem demanded a simply connected region. This restriction may easily be relaxed by the creation of a barrier, a contour line. Consider the multiply connected region of Fig.1.6 In which $f(z)$ is not defined for the interior R'



1.6 Fig.

Cauchy's int contour C, but we can construct a C' for which the theorem holds. If line segments DE and GA arbitrarily close together, then

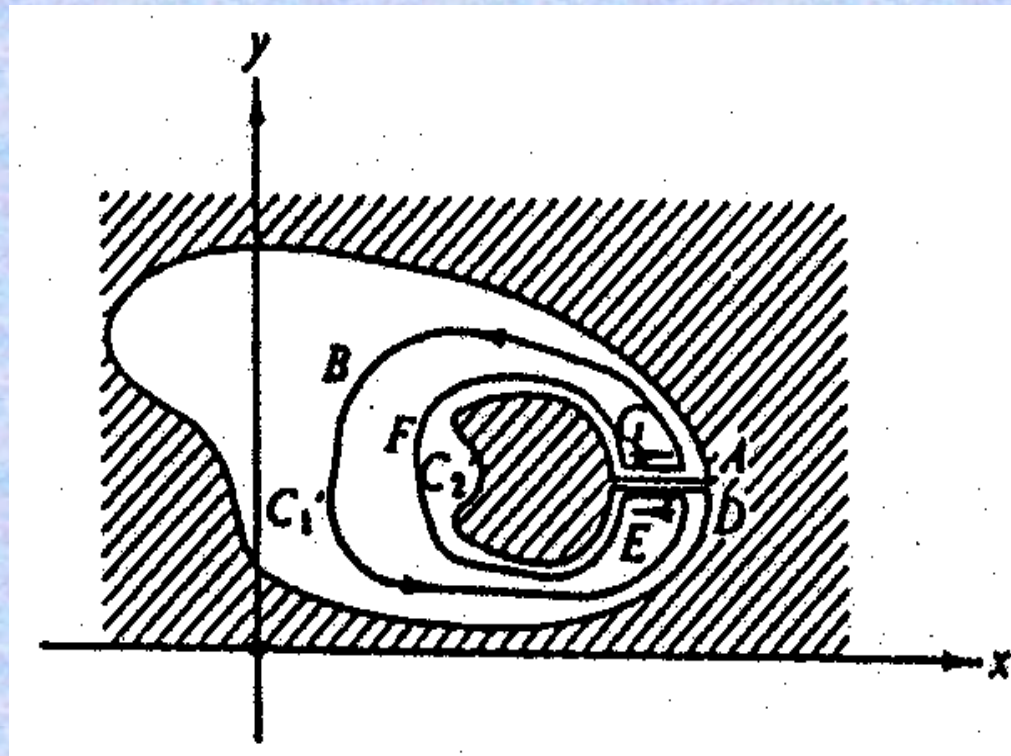
$$\int_G^A f(z)dz = -\int_D^E f(z)dz$$

$$\oint_{\substack{C' \\ (ABDEFGA)}} f(z) dz = \left[\int_{ABD} + \int_{DE} + \int_{GA} + \int_{EFG} \right] f(z) dz$$

$$= \left[\int_{ABD} + \int_{EFG} \right] f(z) dz = 0$$

$$\oint_{C_1'} f(z) dz = \oint_{C_2'} f(z) dz$$

$$ABD \rightarrow C_1' \quad EFG \rightarrow -C_2'$$



Cauchy's Integral Formula

Cauchy's integral formula: If $f(z)$ is analytic on and within a closed contour C then

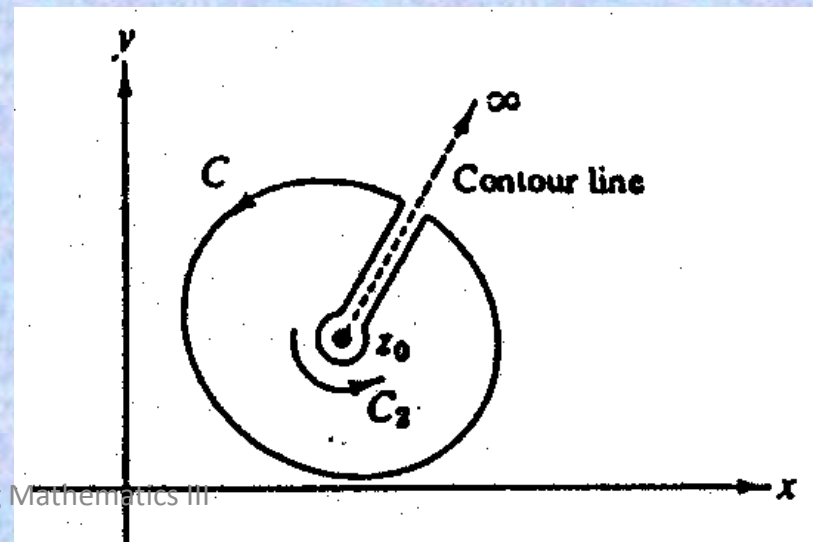
$$\oint_C \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0)$$

in which z_0 is some point in the interior region bounded by C . Note that here $z - z_0 \neq 0$ and the integral is well defined.

Although $f(z)$ is assumed analytic, the integrand $(f(z)/z - z_0)$ is not analytic at $z = z_0$ unless $f(z_0) = 0$. If the contour is deformed as in Fig.1.8 Cauchy's integral theorem applies.

So we have

$$\oint_C \frac{f(z)dz}{z - z_0} - \oint_{C_2} \frac{f(z)dz}{z - z_0} = 0$$



Let $z - z_0 = re^{i\theta}$, here r is small and will eventually be made to approach zero

$$\oint_{C_2} \frac{f(z) dz}{z - z_0} = \oint_{C_2} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta$$

$$(r \rightarrow 0) \quad = if(z_0) \oint_{C_2} d\theta = 2\pi if(z_0)$$

Here is a remarkable result. The value of an analytic function is given at an interior point at $z=z_0$ once the values on the boundary C are specified.

What happens if z_0 is exterior to C ?

In this case the entire integral is analytic on and within C , so the integral vanishes.

$$\frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0} = \begin{cases} f(z_0), & z_0 \text{ interior} \\ 0, & z_0 \text{ exterior} \end{cases}$$

Derivatives

Cauchy's integral formula may be used to obtain an expression for the derivation of $f(z)$

$$\begin{aligned} f'(z_0) &= \frac{d}{dz_0} \left(\frac{1}{2\pi i} \oint \frac{f(z)dz}{z - z_0} \right) \\ &= \frac{1}{2\pi i} \oint f(z)dz \frac{d}{dz_0} \left(\frac{1}{z - z_0} \right) = \frac{1}{2\pi i} \oint \frac{f(z)dz}{(z - z_0)^2} \end{aligned}$$

Moreover, for the n -th order of derivative

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)dz}{(z - z_0)^{n+1}}$$

We now see that, the requirement that $f(z)$ be analytic not only guarantees a first derivative but derivatives of all orders as well! The derivatives of $f(z)$ are automatically analytic. Here, it is worth to indicate that the converse of Cauchy's integral theorem holds as well

Examples

If $f(z) = \sum_{n \geq 0} a_n z^n$ is analytic on and within a circle about the origin, find a_n .

$$f^{(j)}(z) = j! a_j + \sum_{n-j \geq 1} a_n \{ \} z^{n-j}$$

$$f^{(j)}(0) = j! a_j$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint \frac{f(z) dz}{z^{n+1}}$$