DERIVATIVE OF ANALYTIC FUNCTION

Derivatives of Analytic Functions

$z = x + iy$

Let $f(z)$ be analytic around z, then

$$\frac{df(x)}{dx} = g(x) \quad \rightarrow \quad \frac{df(z)}{dz} = g(z)$$

Proof :

$f(z)$ analytic $\rightarrow$ $$f'(z) = \frac{\partial f(x+iy)}{\partial x} = \left.\frac{df(x)}{dx}\right|_{x=z} = g(z)$$

E.g. $$\frac{dx^n}{dx} = n\,x^{n-1} \quad \rightarrow \quad \frac{dz^n}{dz} = n\,z^{n-1}$$

$\therefore$ Analytic functions can be defined by Taylor series of the same coefficients as their real counterparts.

Derivative of Logarithm

CRCs $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

$r=\sqrt{x^2+y^2}$ $\theta=\tan^{-1}\dfrac{y}{x}$

$$\frac{d\ln z}{dz}=\frac{1}{z}$$ for z within each branch.

Proof : $\ln z=\ln r+i\left(\theta+2\pi n\right)=u+iv$ $\rightarrow$ $u=\ln r$, $v=\theta+2\pi n$

$$\frac{\partial u}{\partial x}=\frac{1}{r}\frac{\partial r}{\partial x}=\frac{x}{r^2}=\frac{\partial v}{\partial y}$$

$$\frac{\partial \theta}{\partial y}=\left(1+\frac{y^2}{x^2}\right)^{-1}\frac{1}{x}=\frac{x}{r^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{r}\frac{\partial r}{\partial y}=\frac{y}{r^2}=-\frac{\partial v}{\partial x}$$

$$\frac{\partial \theta}{\partial x}=\left(1+\frac{y^2}{x^2}\right)^{-1}\left(-\frac{y}{x^2}\right)=-\frac{y}{r^2}$$

$\rightarrow$ $\ln z$ is analytic within each branch.

$\therefore$ $$\frac{d\ln z}{dz}=\frac{\partial \ln z}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{x}{r^2}-i\frac{y}{r^2}=\frac{1}{x+iy}=\frac{1}{z}$$

$r^2=z\,z^*$

QED

Derivatives of Functions w(t)

- Consider derivatives of complex-valued functions *w* of real variable *t*

$$w(t) = u(t) + iv(t)$$

where the function *u* and *v* are real-valued functions of *t*. The derivative

$$w'(t), or \frac{d}{dt} w(t)$$

of the function w(t) at a point *t* is defined as

$$w'(t) = u'(t) + iv'(t)$$

Derivatives of Functions w(t)

- Properties

For any complex constant $z_0=x_0+iy_0$,

$$\frac{d}{dt}[z_0 w(t)] = [(x_0 + iy_0)(u+iv)]' = [\underbrace{(x_0 u - y_0 v)}_{u(t)} + i\underbrace{(y_0 u + x_0 v)}_{v(t)}]'$$

$$= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)'$$

$$= (x_0 u' - y_0 v') + i(y_0 u' + x_0 v')$$

$$= (x_0 + iy_0)(u' + iv') = z_0 w'(t)$$

Derivatives of Functions w(t)

- Properties

$$\frac{d}{dt}e^{z_0 t} = z_0 e^{z_0 t}$$

where $z_0 = x_0 + iy_0$. We write

u(t) v(t)

$$e^{z_0 t} = e^{(x_0 + iy_0)t} = e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t$$

$$\frac{d}{dt}e^{z_0 t} = (e^{x_0 t}\cos y_0 t)' + i(e^{x_0 t}\sin y_0 t)'$$

Similar rules from calculus and some simple algebra then lead us to the expression

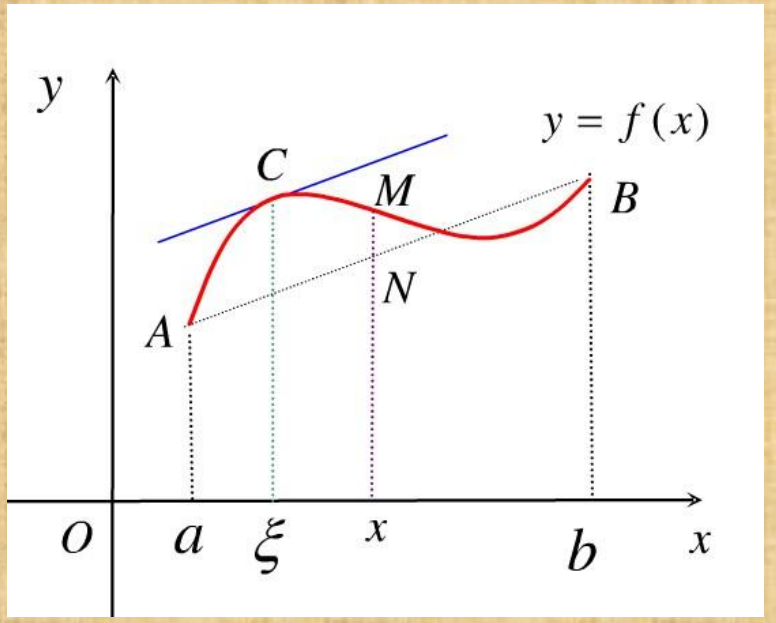
$$\frac{d}{dt}e^{z_0 t} = (x_0 + iy_0)e^{(x_0 + iy_0)t} = z_0 e^{z_0 t}$$

Derivatives of Functions w(t)

- Example

  Suppose that the ***real function*** f(t) is continuous on an interval a≤ t ≤b, if f'(t) exists when a<t<b, the mean value theorem for derivatives tells us that there is a number ζ in the interval a<ζ<b such that

$$f'(\zeta) = \frac{f(b) - f(a)}{b - a}$$



Derivatives of Functions w(t)

- Example (Cont')

The mean value theorem no longer applies for some ***complex functions***. For instance, the function

$$w(t) = e^{it}$$

on the interval $0 \le t \le 2\pi$.

Please note that

$$|w'(t)| = |ie^{it}| = 1$$

And this means that the derivative w'(t) is never zero, while

$$w(2\pi) - w(0) = 0 \Rightarrow w'(\varsigma) \neq \frac{w(2\pi) - w(0)}{2\pi - 0} = 0, \forall \varsigma \in (0, 2\pi)$$

Note: not every rules from calculus holds for complex functions