Evaluation of Real Integrals

Complex variables

Integrals of sinusoidal functions

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta \quad \text{set } z = \exp i\theta \text{ in unit circle}$$

$$\Rightarrow \cos\theta = \frac{1}{2}(z + \frac{1}{z}), \quad \sin\theta = \frac{1}{2i}(z - \frac{1}{z}), \; d\theta = -iz^{-1}dz$$

$$\text{Ex : Evaluate } I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta} d\theta \quad \text{for } b > a > 0$$

$$\cos n\theta = \frac{1}{2}(z^n + z^{-n}) \Rightarrow \cos 2\theta = \frac{1}{2}(z^2 + z^{-2})$$

$$\frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta} d\theta = \frac{\frac{1}{2}(z^2 + z^{-2})(-iz^{-1})dz}{a^2 + b^2 - 2ab \cdot \frac{1}{2}(z + z^{-1})} = \frac{-\frac{1}{2}(z^4 + 1)idz}{z^2(za^2 + zb^2 - abz^2 - ab)}$$

$$= \frac{i}{2ab} \frac{(z^4 + 1)dz}{z^2(z^2 - z(\frac{a}{b} - + \frac{b}{a}) + 1)} = \frac{i}{2ab} \frac{(z^4 + 1)}{z^2(z - \frac{a}{b})(z - \frac{b}{a})} dz$$

Complex variables

$$I=\frac{i}{2ab}\oint_C \frac{z^4+1}{z^2(z-\frac{a}{b})(z-\frac{b}{a})}dz \text{ double poles at } z=0 \text{ and } z=a/b \text{ within the unit circle}$$

$$\text{Residue: } R(z_0)=\lim_{z\to z_0}\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$

(1) pole at $z=0, m=2$

$$R(0)=\lim_{z\to 0}\{\frac{1}{1!}\frac{d}{dz}[z^2\frac{z^4+1}{z^2(z-a/b)(z-b/a)}]\}$$

$$=\lim_{z\to 0}\{\frac{4z^3}{(z-a/b)(z-b/a)}+\frac{(z^4+1)(-1)[2z-(a/b+b/a)]}{(z-a/b)^2(z-b/a)^2}\}=a/b+b/a$$

(2) pole at $z=a/b, m=1$

$$R(a/b)=\lim_{z\to a/b}[(z-a/b)\frac{z^4+1}{z^2(z-a/b)(z-b/a)}]=\frac{(a/b)^4+1}{(a/b)^2(a/b-b/a)}=\frac{-(a^4+b^4)}{ab(b^2-a^2)}$$

$$I=2\pi i\times\frac{i}{2ab}[\frac{a^2+b^2}{ab}-\frac{a^4+b^4}{ab(b^2-a^2)}]=\frac{2\pi a^2}{b^2(b^2-a^2)}$$

Complex variables

Some infinite integrals

$$\int_{-\infty}^{\infty} f(x)dx$$

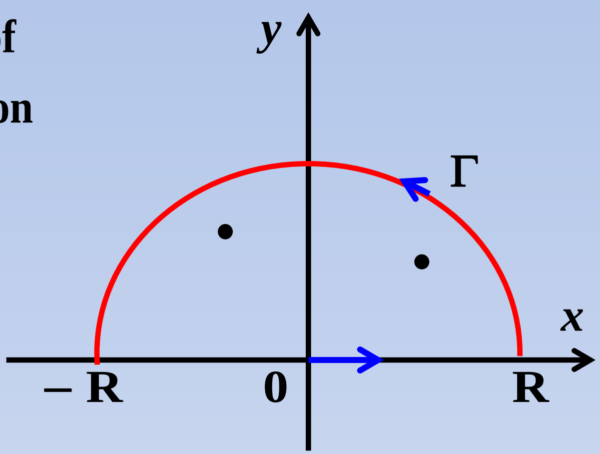
$f(z)$ has the following properties:

(1) $f(z)$ is analytic in the upper half-plane, $\operatorname{Im} z \geq 0$, except for a finite number of poles, none of which is on the real axis.

(2) on a semicircle Γ of radius R, R times the maximum of $|f|$ on Γ tends to zero as $R \to \infty$ (a sufficient condition is that $zf(z) \to 0$ as $|z| \to \infty$).

(3) $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{\infty} f(x)dx$ both exist

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_j R_j$$

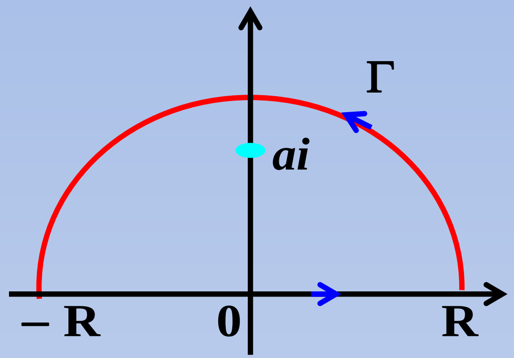


for $\left|\int_{\Gamma} f(z)dz\right| \leq 2\pi R \times$ (maximum of $|f|$ on Γ), the integral along Γ tends to zero as $R \to \infty$.

Complex variables

Ex : Evaluate $I = \int_0^{\infty} \frac{dx}{(x^2+a^2)^4}$ a **is real**

$$\oint_C \frac{dz}{(z^2+a^2)^4} = \int_{-R}^{R} \frac{dx}{(x^2+a^2)^4} + \int_{\Gamma} \frac{dz}{(z^2+a^2)^4} \text{ as } R \to \infty$$



$$\Rightarrow \int_{\Gamma} \frac{dz}{(z^2+a^2)^4} \to 0 \Rightarrow \oint_C \frac{dz}{(z^2+a^2)^4} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^4}$$

$(z^2+a^2)^4 = 0 \Rightarrow$ **poles of order 4 at** $z = \pm ai$,

only $z = ai$ **at the upper half - plane**

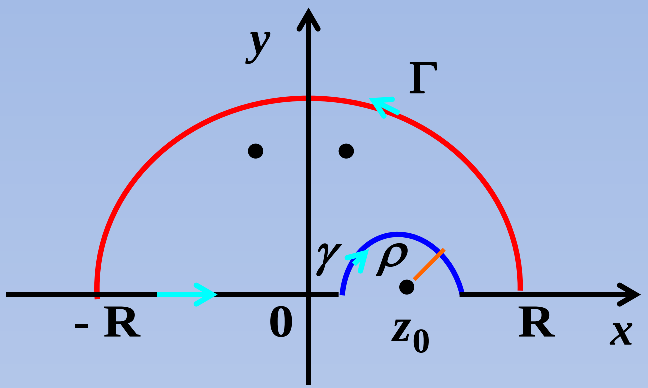
$$\text{set } z = ai + \xi, \xi \to 0 \Rightarrow \frac{1}{(z^2+a^2)^4} = \frac{1}{(2ai\xi + \xi^2)^4} = \frac{1}{(2ai\xi)^4}\left(1 - \frac{i\xi}{2a}\right)^{-4}$$

$$\text{the coefficient of } \xi^{-1} \text{ is } \frac{1}{(2a)^4} \frac{(-4)(-5)(-6)}{3!} \left(\frac{-i}{2a}\right)^3 = \frac{-5i}{32a^7}$$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^4} = 2\pi i\left(\frac{-5i}{32a^7}\right) = \frac{10\pi}{32a^7} \Rightarrow I = \frac{1}{2} \times \frac{10\pi}{32a^7} = \frac{5\pi}{32a^7}$$

Complex variables

For poles on the real axis:



Principal value of the integral, defined as $\rho \to 0$

$$P\int_{-R}^{R} f(x)dx = \int_{-R}^{z_0-\rho} f(x)dx + \int_{z_0+\rho}^{R} f(x)dx$$

for a closed contour C

$$\oint_C f(z)dz = \int_{-R}^{z_0-\rho} f(x)dx + \int_\gamma f(z)dz + \int_{z_0+\rho}^{R} f(x)dx + \int_\Gamma f(z)dz$$

$$= P\int_{-R}^{R} f(x)dx + \int_\gamma f(z)dz + \int_\Gamma f(z)dz$$

(1) for $\int_\gamma f(z)dz$ has a pole at $z = z_0 \Rightarrow \int_\gamma f(z)dz = -\pi i a_1$

(2) for $\int_\Gamma f(z)dz$ set $z = \mathrm{R}e^{i\theta}$ $dz = i\,\mathrm{R}e^{i\theta}\,d\theta$

$$\Rightarrow \int_\Gamma f(z)dz = \int_\Gamma f(\mathrm{R}e^{i\theta})i\,\mathrm{R}e^{i\theta}\,d\theta$$

If $f(z)$ vanishes faster than $1/R^2$ as $R \to \infty$, the integral is zero

Complex variables

Jordan's lemma

(1) $f(z)$ is analytic in the upper half - plane except for a finite number of poles in $\text{Im } z > 0$

(2) the maximum of $|f(z)| \to 0$ as $|z| \to \infty$ in the upper half - plane

(3) $m > 0$, then

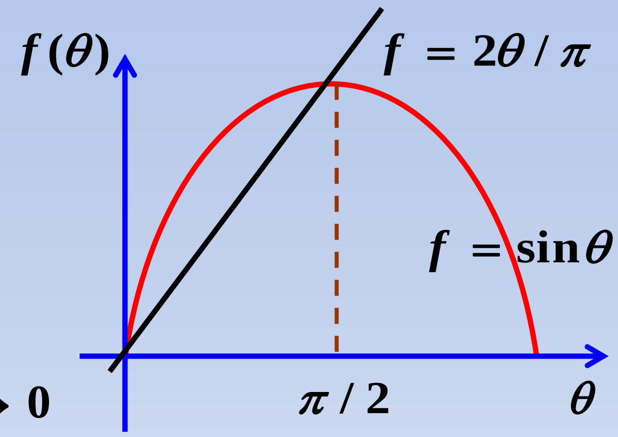
$$I_\Gamma = \int_\Gamma e^{imz} f(z)dz \to 0 \text{ as } R \to \infty, \Gamma \text{ is the semicircular contour}$$

for $0 \le \theta \le \pi/2, \quad 1 \ge \sin\theta/\theta \ge \pi/2$

$|\exp(imz)| = |\exp(-mR\sin\theta)|$

$$I_\Gamma \le \int_\Gamma |e^{imz} f(z)||dz| \le MR\int_0^\pi e^{-mR\sin\theta} d\theta$$

$$= 2MR\int_0^{\pi/2} e^{-mR\sin\theta} d\theta$$



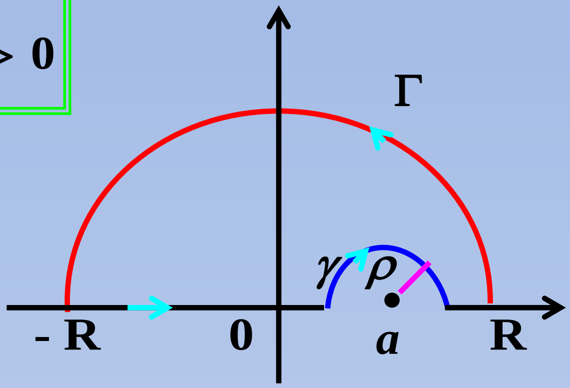
M is the maximum of $|f(z)|$ on $|z| = R$, $R \to \infty$ $M \to 0$

$$I_\Gamma < 2MR\int_0^{\pi/2} e^{-mR(2\theta/\pi)} d\theta = \frac{\pi M}{m}(1 - e^{-mR}) < \frac{\pi M}{m}$$

as $R \to \infty \Rightarrow M \to 0 \Rightarrow I_\Gamma \to 0$

Complex variables

Ex : Find the principal value of $\int_{-\infty}^{\infty} \frac{\cos mx}{x-a}dx$ a **real,** $m > 0$



Consider the integral $I = \oint_C \frac{e^{imz}}{z-a}dz = 0$ no pole in the upper half - plane, and $|(z-a)^{-1}| \to 0$ as $|z| \to \infty$

$$I = \oint_C \frac{e^{imz}}{z-a}dz$$

$$= \int_{-R}^{a-\rho} \frac{e^{imx}}{x-a}dx + \oint_\gamma \frac{e^{imz}}{z-a}dz + \int_{a+\rho}^{R} \frac{e^{imx}}{x-a}dx + \int_\Gamma \frac{e^{imz}}{z-a}dz = 0$$

As $R \to \infty$ and $\rho \to 0 \Rightarrow \int_\Gamma \frac{e^{imz}}{z-a}dz \to 0$

$$\Rightarrow P\int_{-\infty}^{\infty} \frac{e^{imx}}{x-a}dx - i\pi a_{-1} = 0 \text{ and } a_{-1} = e^{ima}$$

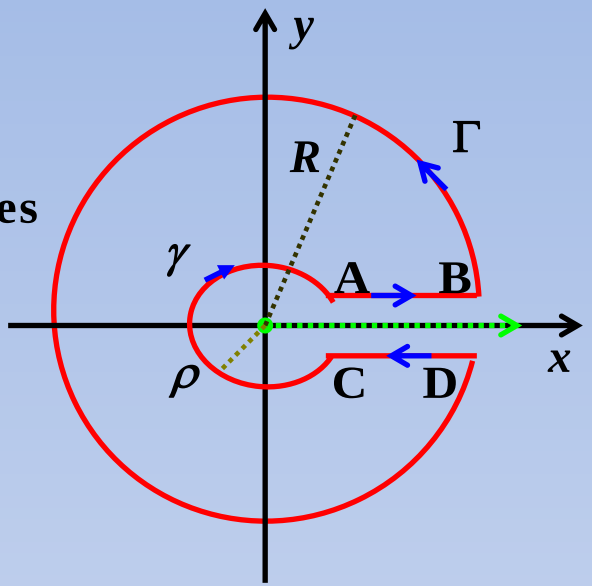
$$\Rightarrow P\int_{-\infty}^{\infty} \frac{\cos mx}{x-a}dx = -\pi \sin ma \text{ and } P\int_{-\infty}^{\infty} \frac{\sin mx}{x-a}dx = \pi \cos ma$$

Complex variables

Integral of multivalued functions

Multivalued functions such as $z^{1/2}$, Lnz

Single branch point is at the otigin. We let $R \to \infty$ and $\rho \to 0$. The integrand is multivalued, its values along two lines AB and CD joining $z = \rho$ to $z = R$ are not equal and opposite.



$$\text{Ex}: I = \int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}} \quad \text{for } a > 0$$

(1) the integrand $f(z) = (z+a)^{-3} z^{-1/2}$, $|zf(z)| \to 0$ as $\rho \to 0$ and $R \to \infty$

the two circles make no contribution to the contour integral

(2) pole at $z = -a$, and $(-a)^{1/2} = a^{1/2} e^{i\pi/2} = ia^{1/2}$

$$R(-a) = \lim_{z \to -a} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} [(z+a)^3 \frac{1}{(z+a)^3 z^{1/2}}]$$

$$= \lim_{z \to -a} \frac{1}{2!} \frac{d^2}{dz^2} z^{-1/2} = \frac{-3i}{8a^{5/2}}$$

Complex variables

$$\int_{AB} dz + \int_{\Gamma} dz + \int_{DC} dz + \int_{\gamma} dz = 2\pi i(\frac{-3i}{8a^{5/2}})$$

$$\text{and} \int_{\Gamma} dz = 0 \text{ and } \int_{\gamma} dz = 0$$

$$\text{along line AB} \Rightarrow z = xe^{i0}, \text{ along line CD} \Rightarrow z = xe^{i2\pi}$$

$$\int_{0, A\to B}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} + \int_{\infty, C\to D}^{0} \frac{dx}{(xe^{i2\pi} + a)^3 x^{1/2} e^{(1/2\times 2\pi i)}} = \frac{3\pi}{4a^{5/2}}$$

$$\Rightarrow (1 - \frac{1}{e^{i\pi}})\int_0^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{4a^{5/2}}$$

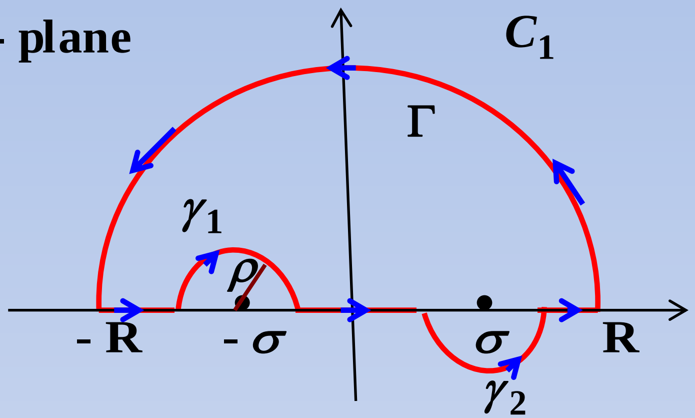
$$\Rightarrow \int_0^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{8a^{5/2}}$$

Complex variables

Ex : Evaluate $I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2} dx$

$$\oint_C \frac{z \sin z}{z^2 - \sigma^2} dz = \frac{1}{2i} \oint_{C_1} \frac{z e^{iz}}{z^2 - \sigma^2} dz - \frac{1}{2i} \oint_{C_2} \frac{z e^{-iz}}{z^2 - \sigma^2} dz = I_1 + I_2$$

(1) for I_1, the contour is choosed on the upper half - plane due to the term e^{iz}, and only one pole at $z = \sigma$.



$$I_1 = \frac{1}{2i} \oint_{C_1} \frac{z e^{iz}}{z^2 - \sigma^2} dz = \frac{1}{2i} \int_{-R}^{-\sigma-\rho} \frac{x e^{ix}}{x^2 - \sigma^2} dx$$

$$+ \frac{1}{2i} \int_{-\sigma+\rho}^{\sigma-\rho} \frac{x e^{ix}}{x^2 - \sigma^2} dx + \frac{1}{2i} \int_{\sigma+\rho}^{\infty} \frac{x e^{ix}}{x^2 - \sigma^2} dx$$

$$+ \frac{1}{2i} \int_{\gamma_1} \frac{z e^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\gamma_2} \frac{z e^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\Gamma} \frac{z e^{iz}}{z^2 - \sigma^2} dz$$

$$= \frac{1}{2i} 2\pi i \times \mathrm{Res}(z = \sigma) = \pi \frac{\sigma e^{i\sigma}}{2\sigma} = \frac{\pi}{2} e^{i\sigma}$$

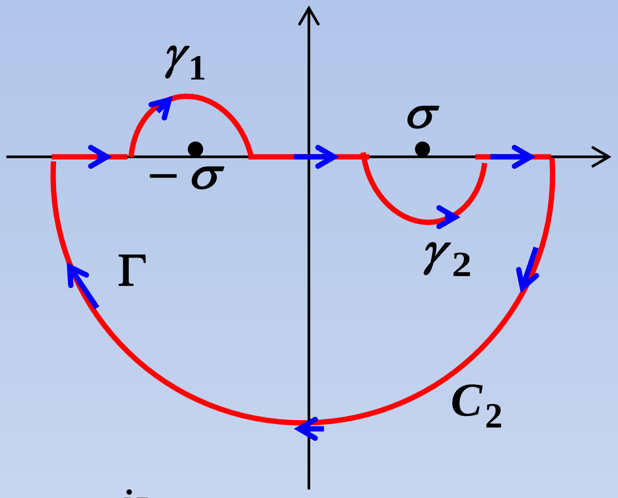
Complex variables

As $\rho \to 0$ and $R \to \infty \Rightarrow \int_{\Gamma} dz \to 0$

$$\frac{1}{2i}\int_{\gamma_1}\frac{ze^{iz}}{(z+\sigma)(z-\sigma)}dz = \frac{1}{2i}\times(-\pi i)\text{Res}(z=-\sigma) = \frac{-\pi}{4}e^{-i\sigma}$$

$$\frac{1}{2i}\int_{\gamma_2}\frac{ze^{iz}}{(z+\sigma)(z-\sigma)}dz = \frac{1}{2i}\times \pi i\text{Res}(z=\sigma) = \frac{\pi}{4}e^{i\sigma}$$

$$I_1 = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2-\sigma^2}dx + \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) = \frac{\pi}{2}e^{i\sigma}$$



(2) for I_2, we choose the lower half - plane by the

term e^{-iz}, only one pole at $z = -\sigma$

$$I_2 = \frac{-1}{2i}\oint_{C_2}\frac{ze^{-iz}}{z^2-\sigma^2}dz = \frac{-1}{2i}\int_{-R}^{-\sigma-\rho}\frac{xe^{-ix}}{x^2-\sigma^2}dx$$

$$-\frac{1}{2i}\int_{-\sigma+\rho}^{\sigma-\rho}\frac{xe^{-ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{\sigma+\rho}^{\infty}\frac{xe^{-ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{\gamma_1}\frac{ze^{-iz}}{z^2-\sigma^2}dz$$

$$-\frac{1}{2i}\int_{\gamma_2}\frac{ze^{-iz}}{z^2-\sigma^2}dz - \frac{1}{2i}\int_{\Gamma}\frac{ze^{-iz}}{z^2-\sigma^2}dz = \left(\frac{-1}{2i}\right)\times(-2\pi i)\frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{2}e^{i\sigma}$$

Complex variables

$$\text{As } \rho \to 0, R \to \infty \Rightarrow \int_{\Gamma} dz \to 0$$

$$\frac{-1}{2i}\int_{\gamma_1} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)}dz = (\frac{-1}{2i})(-\pi i)\frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{4}e^{i\sigma}$$

$$\frac{-1}{2i}\int_{\gamma_2} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)}dz = (\frac{-1}{2i})(\pi i)\frac{\sigma e^{-i\sigma}}{2\sigma} = \frac{-\pi}{4}e^{-i\sigma}$$

$$I_2 = \frac{-1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx + \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) = \frac{\pi}{2}e^{i\sigma}$$

$$\Rightarrow \frac{-1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx = \frac{\pi}{2}e^{i\sigma} - \frac{1}{4}(e^{i\sigma} - e^{-i\sigma})$$

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2-\sigma^2}dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx$$

$$= \frac{\pi}{2}e^{i\sigma} - \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) + \frac{\pi}{2}e^{i\sigma} - \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma})$$

$$= \pi e^{i\sigma} - \frac{\pi}{2}e^{i\sigma} + \frac{\pi}{2}e^{-i\sigma} = \frac{\pi}{2}(e^{i\sigma} + e^{-i\sigma}) = \pi\cos\sigma$$