Harmonic Function

Harmonic Functions

- A Harmonic Function

A ***real-valued*** function H of ***two real variables*** x and y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0$$

Known as Laplace's equation.

Harmonic Functions

- Theorem 1

If a function f (z) = u(x, y) + iv(x, y) is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof: $f(z) = u(x,y) + iv(x,y)$ is analytic in D

$$\Rightarrow \quad u_x = v_y \;\&\; u_y = -v_x$$

Differentiating both sizes of these equations with respect to x and y respectively, we have

$$\begin{matrix} u_{xx} = v_{yx} \;\&\; u_{yx} = -v_{xx} \\ u_{xy} = v_{yy} \;\&\; u_{yy} = -v_{xy} \end{matrix} \quad \xrightarrow{\text{continuity}} \quad \begin{matrix} u_{xx} = v_{xy} \;\&\; u_{xy} = -v_{xx} \\ u_{xy} = v_{yy} \;\&\; u_{yy} = -v_{xy} \end{matrix}$$

$$u_{xx} + u_{yy} = 0 \;\&\; v_{xx} + v_{yy} = 0$$

Theorem in Sec.52:
a function is analytic at a point, then its real and imaginary components have continuous partial derivatives of all order at that point.

Harmonic Functions

- Example 3

The function $f(z)=i/z^2$ is analytic whenever $z \neq 0$ and since

$$\frac{i}{z^2}=\frac{i\times(\bar{z})^2}{z^2\times(\bar{z})^2}=\frac{2xy+i(x^2-y^2)}{(x^2+y^2)^2}$$

The two functions

$$u(x,y)=\frac{2xy}{(x^2+y^2)^2} \qquad v(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

Harmonic Functions

- Harmonic conjugate

If ***two*** given function u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equation throughout D, then v is said to be a harmonic conjugate of u.

Is the definition symmetry for u and v?

Cauchy-Riemann equation $$u_x = v_y \ \& \ u_y = -v_x$$

If u is a harmonic conjugate of v, then

$$u_x = -v_y \ \& \ u_y = v_x$$

Harmonic Functions

- Theorem 2

A function f (z) = u(x, y) + iv(x, y) is analytic in a domain D if and only if v is a harmonic conjugate of u.

- Example 4

The function $f(z) = z^2$ is entire function, and its real and imaginary components are $u(x, y) = x^2 - y^2$ & $v(x, y) = 2xy$

Based on the Theorem 2, v is a harmonic conjugate of u throughout the plane. However, u is not the harmonic conjugate of v, since $g(z) = 2xy + i(x^2 - y^2)$ is not an analytic function.

Harmonic Functions

- Example 5

Obtain a harmonic conjugate of a given function.

$$u(x,y) = y^3 - 3x^2 y$$

Suppose that v is the harmonic conjugate of the given function

Then $\quad u_x = v_y \;\&\; u_y = -v_x$

$$u_x = -6xy = v_y \quad \Rightarrow \quad v = -3xy^2 + \phi(x)$$

$$u_y = 3y^2 - 3x^2 = -v_x \quad \Rightarrow \quad 3y^2 - 3x^2 = -(-3y^2 + \phi'(x))$$

$$\phi'(x) = 3x^2 \Rightarrow \phi(x) = x^3 + C$$

$$v = -3xy^2 + x^3 + C$$