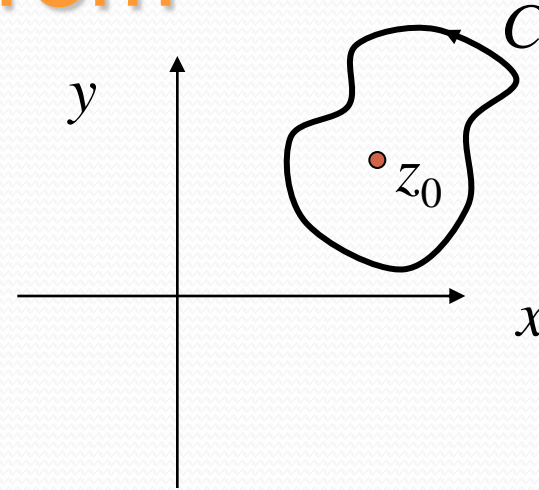# Residue Theorem

# The Residue Theorem

**Consider a line integral about a path enclosing an isolated singular point:**

$$\oint_C f(z)\,dz$$

**Expand *f(z)* in a Laurent series, deform the contour *C* to a circle of radius *r* centered at $z_0$, and evaluate the integral:**

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Let $z - z_0 \equiv re^{i\theta}, \quad dz = rie^{i\theta}d\theta,$

$$\Rightarrow \oint_C f(z)\,dz = i \sum_{n=-\infty}^{\infty} a_n r^{n+1} \underbrace{\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta}_{= 0,\, n \neq -1}$$

$$= 2\pi i\, a_{-1}$$

# The Residue Theorem (cont.)

The value $a_{-1}$ corresponding to an isolated singular point $z_0$ is called the "residue of $f(z)$ at $z_0$".

$$\oint_C f(z)\,dz \;=\; 2\pi i \operatorname{Res} f(z_0)$$

Note path orientation is assumed counterclockwise

# The Residue Theorem (cont.)

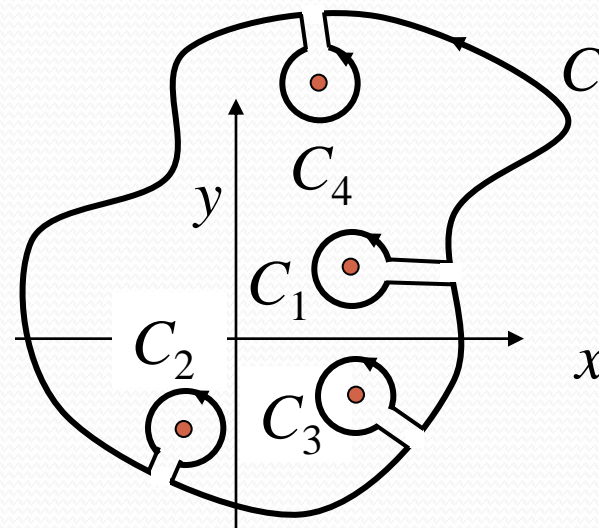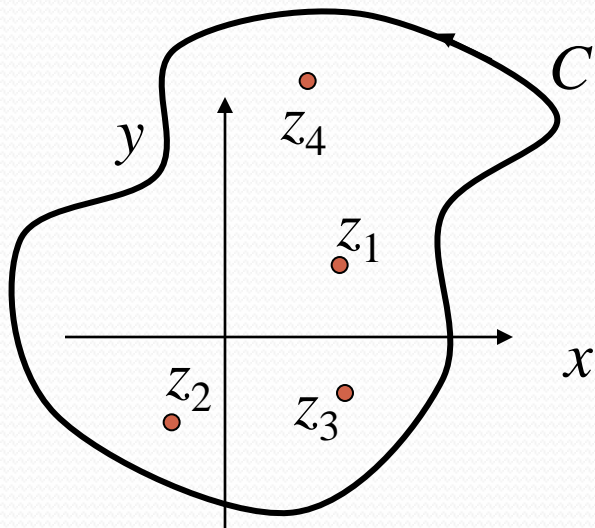**Extend the theorem to multiple isolated singularities in the usual way:**

$$\oint_{C-\sum_n C_n} f(z)\,dz \;=\; \oint_C f(z)\,dz - \sum_n \oint_{C_n} f(z)\,dz = 0$$

$$\Rightarrow \oint_C f(z)\,dz \;=\; \sum_n \oint_{C_n} f(z)\,dz$$
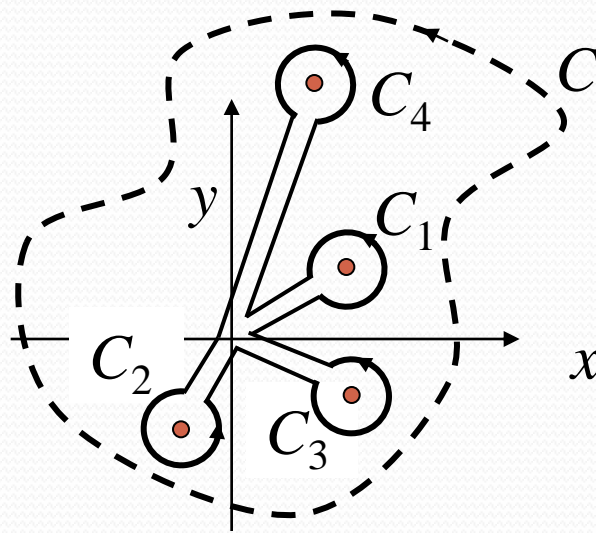
$$\oint_C f(z)\,dz \;=\; 2\pi i \sum_n \operatorname{Res} f(z_n)$$

**Isolated singularities at $z_n$**

$$= 2\pi i \;\times\; \text{sum of residues enclosed by } C$$

# The Residue Theorem (cont.)

**Alternatively, shrink the path to a point, leaving only the singularities encircled:**



**Isolated singularities at $z_n$**

$$\oint_C f(z)\,dz \; = \; 2\pi i \sum_n \operatorname{Res} f(z_n)$$

$$= 2\pi i \; \times \; \text{sum of residues enclosed by } C$$

# The Residue Theorem (cont.)

**Note the residue theorem subsumes several of our earlier results and theorems, e.g. :**

**Cauchy's Formula:**

$$\oint_C f(z)\,dz = 0 \text{ if } f(z) \text{ analytic (no isolated singularities) in } C$$

**Cauchy Integral Formula:**

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0) \text{ if } f(z) \text{ analytic}$$

$$\text{(no isolated singularities) in } C$$

# Evaluating Residues

- Construct Laurent series about each singularity $z_n$, identify coefficient $a_{-1,n} = \text{Res}\, f(z_n)$. Sometimes this can be a tedious approach.

- For a *simple* pole at $z = z_0$,

$$\text{Res}\, f(z_0) = \lim_{z \to z_0} \left( z - z_0 \right) f(z)$$

$$= \lim_{z \to z_0} \left( z - z_0 \right) \left( \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \cdots \right)$$

$$= a_{-1}$$

$$\boxed{\text{Res}\, f(z_0) = \lim_{z \to z_0} \left( z - z_0 \right) f(z)}$$

# Evaluating Residues (cont.)

## Example:

- $f(z) = \dfrac{2z}{(z+2)(z^2+1)}$   has *simple* poles at   $z = -2, +i, -i$

$\Rightarrow$

$$\operatorname{Res} f(-2) = \lim_{z \to -2} (z+2) \frac{2z}{(z+2)(z^2+1)} = -\frac{4}{5}$$

$$\operatorname{Res} f(i) = \lim_{z \to i} (z-i) \frac{2z}{(z+2)(z+i)(z-i)} = \frac{2i}{(2+i)\,2i} = \frac{2-i}{5}$$

$$\operatorname{Res} f(-i) = \lim_{z \to -i} (z+i) \frac{2z}{(z+2)(z-i)(z+i)} = \frac{(-2i)}{(2-i)(-2i)} = \frac{2+i}{5}$$

# Evaluating Residues (cont.)

Example:

$$f(z) = \frac{1}{\sin z} \text{ has simple poles at } z = n\pi, \ n = 0, \pm 1, \pm 2, \cdots$$

$$\Rightarrow$$

**L'Hospital's rule**

$$\operatorname{Res} f(n\pi) = \lim_{z \to n\pi} (z - n\pi) \times \frac{1}{\sin z} = \lim_{z \to n\pi} \frac{1}{\cos z} = (-1)^n$$

Alternatively,

$$\operatorname{Res} f(n\pi) = \lim_{z \to n\pi} (z - n\pi) \frac{1}{\sin z} = \lim_{z \to n\pi} (z - n\pi) \frac{1}{\sin(z - n\pi + n\pi)}$$

$$= \lim_{z \to n\pi} (z - n\pi) \frac{1}{\sin(z - n\pi)\cos n\pi + \cos(z - n\pi)\sin n\pi}$$

$$= \lim_{z \to n\pi} \frac{(z - n\pi)}{\sin(z - n\pi)\cos n\pi} = (-1)^n \text{ since we already know } \lim_{z \to 0} \frac{z}{\sin z} \to 1$$

# Evaluating Residues (cont.)

- The above rule can be specialized to functions of the form $g(z)/f(z)$ where $g(z)$ is analytic, non-vanishing, and $f(z)$ has a simple zero, all at $z_0$ :

$$\operatorname{Res} \frac{g(z_0)}{f(z_0)} = \lim_{z \to z_0}(z - z_0)\frac{g(z)}{f(z)} = \lim_{z \to z_0}(z - z_0)\frac{g(z)}{f(z) - f(z_0)}$$

$$= \frac{g(z_0)}{\displaystyle\lim_{z \to z_0}\frac{f(z) - f(z_0)}{(z - z_0)}} = \frac{g(z_0)}{f'(z_0)}$$

$$\Rightarrow \operatorname{Res} \frac{g(z_0)}{f(z_0)} = \frac{g(z_0)}{f'(z_0)}$$

Example:

$$\tan z = \frac{\sin z}{\cos z} \quad \text{has simple poles at} \quad z = \frac{(2n+1)\pi}{2},$$

$$\text{for} \quad n = 0, \pm 1, \pm 2, \cdots$$

$$\Rightarrow \operatorname{Res} \tan\frac{(2n+1)\pi}{2} = \frac{\sin\frac{(2n+1)\pi}{2}}{-\sin\frac{(2n+1)\pi}{2}} = -1$$

# Evaluating Residues (cont.)

- For a pole of finite order $m$ at $z = z_0$:

At the pole, the Laurent series is

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Hence the following series is a Taylor series about the pole,

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots$$

for which the formula for the coefficent of $(z-z_0)^{m-1}$ is

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0}$$

$$\text{Res } f(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0} \qquad (m \text{ finite})$$

# Evaluating Residues (cont.)

$$\text{Res } f(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Bigg|_{z=z_0} \qquad (m \text{ finite})$$

Example:

$$f(z) = \frac{2z^2 + z}{(z+2)^3} \qquad \text{has a pole of order 3 at } z = -2$$

$$\Rightarrow \text{Res } f(-2) = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z+2)^3 \frac{2z^2+z}{(z+2)^3}\right]\Bigg|_{z=-2}$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left[2z^2 + z\right]\Bigg|_{z=-2} = \frac{4}{2} = 2$$

# Evaluating Residues (cont.)

Example: $f(z) = \dfrac{1}{\sin^2 z}$ has poles of order 2 at $z = n\pi, \ n = 0, \pm 1, \pm 2, \cdots$

$$\Rightarrow \operatorname{Res} f(n\pi) = \left. \frac{d}{dz}\left[ \frac{(z-n\pi)^2}{\sin^2 z} \right] \right|_{z=n\pi} = \left. \frac{d}{dz}\left[ \frac{(z-n\pi)^2}{\sin^2(z-n\pi+n\pi)} \right] \right|_{z=n\pi}$$

$$= \left. \frac{d}{dz}\left[ \frac{(z-n\pi)^2}{\left[\sin(z-n\pi)\cos n\pi + \cos(z-n\pi)\sin n\pi\right]^2} \right] \right|_{z=n\pi} = \left. \frac{d}{dz}\left[ \frac{(z-n\pi)^2}{\sin^2(z-n\pi)} \right] \right|_{z=n\pi}$$

**Let** $u = z - n\pi$

$$= \left. \frac{d}{du}\left[ \frac{u^2}{\sin^2 u} \right] \right|_{u=0} = \left. \left[ \frac{2u\left( \sin^{2} u - u\,\sin u\,\cos u \right)}{\sin^{3} u} \right] \right|_{u=0}$$

**L'hospital's rule**

$$= \left. \left[ \frac{2\left( \sin u - u \cos u \right) + 2u^2 \sin u}{3\sin^2 u \cos u} \right] \right|_{u=0}$$

**L'hospital's rule**

$$= \left. \left[ \frac{6u \sin u + 2u^2 \cos u}{3\sin u \left( 2\cos^2 u - \sin^2 u \right)} \right] \right|_{u=0}$$

**L'hospital's rule**

$$= \left. \left[ \frac{6\sin u + 10u \cos u - 2u^2 \sin u}{3\cos u \left( 2\cos^2 u - \sin^2 u \right) - 18\sin^2 u \cos u} \right] \right|_{u=0} = 0$$

# Evaluating Residues (cont.)

The previous result might be "more easily" obtained as follows:

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\sin^2(z - n\pi + n\pi)} = \frac{1}{\sin^2(z - n\pi)} = \frac{1}{\sin^2 u}\bigg|_{u = z - n\pi \to 0}$$

$$= \frac{1}{\left(u - \dfrac{u^3}{3!} + \dfrac{u^5}{5!} - \cdots\right)^2} = \frac{1}{u^2\left(1 - \dfrac{u^2}{3!} + \dfrac{u^4}{5!} - \cdots\right)^2}$$

**Geometric Series**

$$= \frac{1}{u^2}\left[1 + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right) + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right)^2 + \cdots\right]^2$$

$$= \frac{1}{u^2}\left[1 + \frac{u^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)u^4 + \cdots\right]^2 = \frac{1}{u^2} + \frac{2}{3!} + \cdots$$

missing $\dfrac{a_{-1}}{u}$ term $\Rightarrow \operatorname{Res} f(n\pi) = 0$
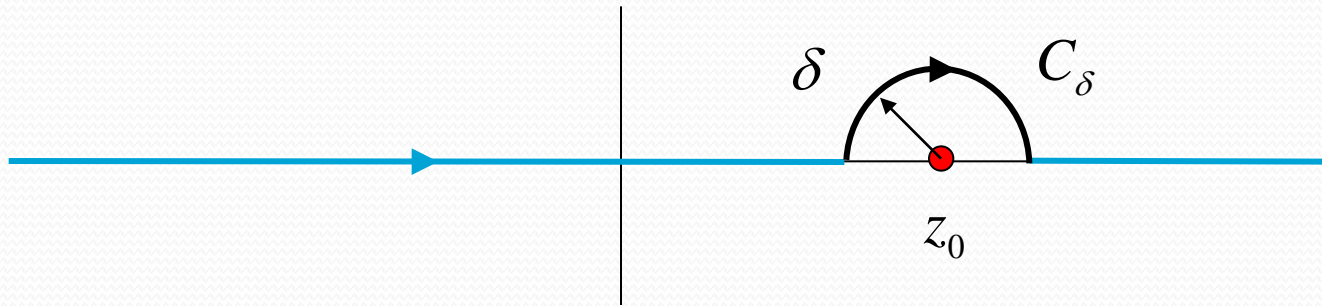
# Evaluating Residues (cont.)

Example:

$$f(z) = e^{\frac{1}{z}} \quad \text{(isolated essential singularity at } z = 0\text{)}$$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \dots$$

Residue: $\boxed{a_{-1} = 1}$

# Evaluating Residues (cont.)

Extension for simple poles (going halfway around)



As $\delta \to 0$:

$$\int_{C_\delta} f(z)\, dz = -\pi i \operatorname{Res} f(z_0)$$
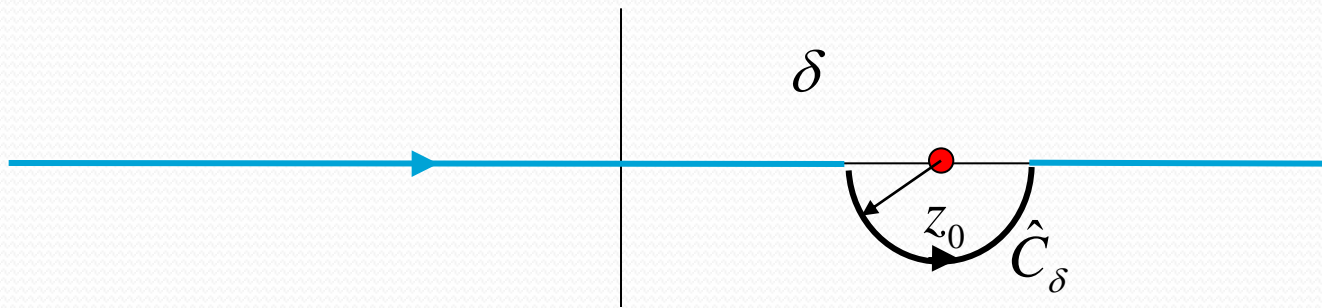
Proof:

$$z - z_0 = \delta e^{i\theta}$$

$$\int_{C_\delta} \frac{a_{-1}}{z - z_0}\, dz = a_{-1} \int_{C_\delta} \frac{dz}{z - z_0} = a_{-1} \int_\pi^0 \frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\, d\theta = a_{-1}\left(-\pi i\right)$$

# Evaluating Residues (cont.)

What is the result if the contour goes around the simple pole on the opposite side?



Hint: The contour $\hat{C}_\delta - C_\delta$ encircles the pole in the counterclockwise direction so that

$$\oint_{\hat{C}_\delta - C_\delta} f(z)\,dz = 2\pi i \operatorname{Res} f(z_0)$$

$$\Rightarrow \int_{\hat{C}_\delta} f(z)\,dz = \pi i \operatorname{Res} f(z_0)$$

# Evaluation of residues at multiple poles

If $f(z)$ has a pole of order n at $z = a$ and no other singularity, $f(z)$ is:

$$f(z) = \frac{F(z)}{(z-a)^n}$$

where $n$ is a finite integer, and $F(z)$ is analytic at $z = a$.

$F(z)$ can be expanded by the Taylor series:

$$F(z) = F(a) + (z-a)F'(a) + \frac{(z-a)^2}{2!}F''(a) + ... + \frac{(z-a)^n}{n!}F^n(a) + ...$$

Dividing throughout by $(z-a)^n$

$$f(z) = \frac{F(a)}{(z-a)^n} + \frac{F'(a)}{(z-a)^{n-1}} + ... + \frac{F^{n-1}(a)}{(n-1)!(z-a)} + ...$$

The residue at z = a is the coefficient of $(z-a)^{-1}$

The residue at a pole of order $n$ situated at $z = a$ is:

$$B_1 = \frac{F^{n-1}(a)}{(n-1)!}\bigg|_{z=a} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]_{z=a}$$

**EXAMPLE:**

Evaluate $\int_C \dfrac{\cos 2z}{(z-a)^3} dz$ around a circle of radius $|z| > |a|$.

$\dfrac{\cos 2z}{(z-a)^3}$ has a pole of order 3 at $z = a$, and the residue is:

$$B_1 = \frac{F^{n-1}(a)}{(n-1)!}\bigg|_{z=a} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\Big[(z-a)^n f(z)\Big]_{z=a}$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-a)^3 \frac{\cos 2z}{(z-a)^3}\right]_{z=a} = -2\cos 2a$$

$$\therefore \int_C \frac{\cos 2z}{(z-a)^3}dz = 2\pi i(-2\cos 2a)$$