

Singularities, Zeros And Poles

Singularities

- We have seen that the function $w = z^3$ is analytic everywhere except at $z = \infty$; whilst the function $w = z^{-1}$ is analytic everywhere except at $z = 0$.
- In fact, **NO** function except a **constant** is analytic throughout the complex plane, and every function except of a complex variable has one or more points in the z plane where it ceases to be analytic.
- These points are called “singularities”.

Types of singularities

- Three types of singularities exist:
 - Poles or unessential singularities
 - “single-valued” functions
 - Essential singularities
 - “single-valued” functions
 - Branch points
 - “multivalued” functions

Poles or unessential singularities

- A pole is a point in the complex plane at which the value of a function becomes *infinite*.
- For example, $w = z^{-1}$ is infinite at $z = 0$, and we say that the function $w = z^{-1}$ has a pole at the origin.
- A pole has an “order”:
 - The pole in $w = z^{-1}$ is first order.
 - The pole in $w = z^{-2}$ is second order.

The order of a pole

If $w = f(z)$ becomes infinite at the point $z = a$, we define:

$$g(z) = (z - a)^n f(z) \quad \text{where } n \text{ is an integer.}$$

If it is possible to find a finite value of n which makes $g(z)$ analytic at $z = a$, then, the pole of $f(z)$ has been “removed” in forming $g(z)$.

The order of the pole is defined as the minimum integer value of n for which $g(z)$ is analytic at $z = a$.

$$w = \frac{1}{z} \quad \text{pole, (a=0)}$$

$$(z)^n \frac{1}{z} = g(z) \quad \text{Order} = 1$$

Essential singularities

- Certain functions of complex variables have an infinite number of terms which all approach infinity as the complex variable approaches a specific value. These could be thought of as poles of infinite order, but as the singularity cannot be removed by multiplying the function by a finite factor, they cannot be poles.
- This type of singularity is called an *essential singularity* and is portrayed by functions which can be expanded in a descending power series of the variable.
- Example: $e^{1/z}$ has an essential singularity at $z = 0$.

Essential singularities can be distinguished from poles by the fact that they cannot be removed by multiplying by a factor of finite value.

Example:

$$w = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots + \frac{1}{n!z^n} + \dots \leftarrow \text{infinite at the origin}$$

We try to remove the singularity of the function at the origin by multiplying z^p

$$z^p w = z^p + z^{p-1} + \frac{z^{p-2}}{2!} + \dots + \frac{z^{p-n}}{n!} + \dots$$

It consists of a finite number of positive powers of z , followed by an infinite number of negative powers of z .

All terms are positive

$$\text{As } z \rightarrow 0, \quad z^p w \rightarrow \infty$$

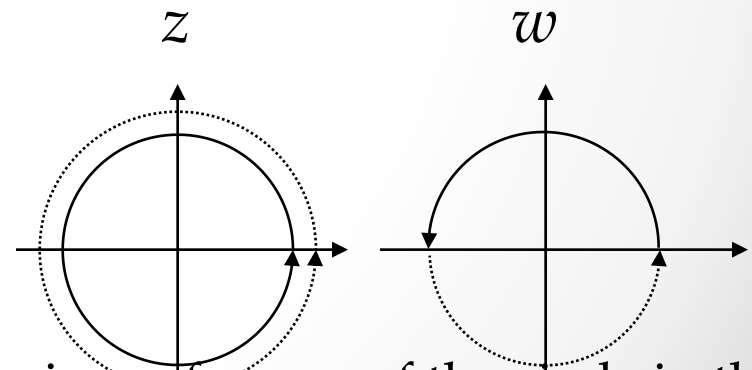
It is impossible to find a finite value of p which will remove the singularity in $e^{1/z}$ at the origin.

The singularity is “essential”.

Branch points

- The singularities described above arise from the non-analytic behaviour of single-valued functions.
- However, multi-valued functions frequently arise in the solution of engineering problems.
- For example:

$$\boxed{w = z^{\frac{1}{2}}} \xrightarrow{z = re^{i\theta}} \boxed{w = r^{\frac{1}{2}} e^{\frac{1}{2}i\theta}}$$



For any value of z represented by a point on the circumference of the circle in the z plane, there will be two corresponding values of w represented by points in the w plane.

$$w = z^{\frac{1}{2}} \quad z = re^{i\theta} \quad w = r^{\frac{1}{2}} e^{\frac{1}{2}i\theta}$$

$$w = u + iv \quad e^{i\theta} = \cos \theta + i \sin \theta$$

$$u = r^{\frac{1}{2}} \cos \frac{1}{2}\theta \quad \text{and} \quad v = r^{\frac{1}{2}} \sin \frac{1}{2}\theta$$

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{2\sqrt{r}} \cos \frac{1}{2}\theta & \frac{\partial v}{\partial r} &= \frac{1}{2\sqrt{r}} \sin \frac{1}{2}\theta \\ \frac{\partial u}{\partial \theta} &= -\frac{1}{2}\sqrt{r} \sin \frac{1}{2}\theta & \frac{\partial v}{\partial \theta} &= \frac{1}{2}\sqrt{r} \cos \frac{1}{2}\theta \end{aligned}$$

when $0 \leq \theta \leq 2\pi$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

A given range, where the function is single valued: the "branch"

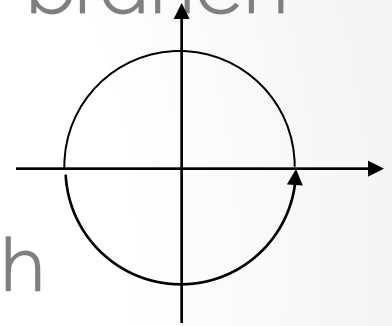
The particular value of z at which the function becomes infinite or zero

is called the "branch point".

The origin is the branch point here

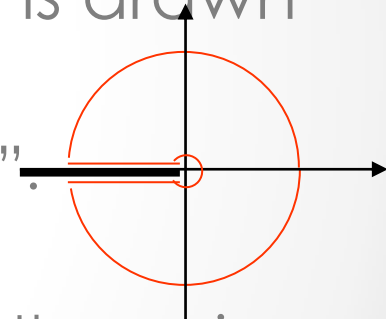
Branch point

- A function is only multi-valued around closed contours which enclose the branch point.
- It is only necessary to eliminate such contours and the function will become single valued.
 - The simplest way of doing this is to erect a barrier from the branch point to infinity and not allow any curve to cross the barrier.
 - The function becomes single valued and analytic for all permitted curves.



Barrier - branch cut

- The barrier must start from the branch point but it can go to infinity in any direction in the z plane, and may be either curved or straight.
- In most normal applications, the barrier is drawn along the negative real axis.
 - The branch is termed the “principle branch”.
 - The barrier is termed the “branch cut”.
 - For the example given in the previous slide, the region, the barrier confines the function to the region in which the argument of z is within the range $-\pi < \theta < \pi$.



Zeros and Poles of order m

Consider a function f that is analytic at a point z_0 .

(From Sec. 40). $f^{(n)}(z)$ ($n=1, 2, \dots$) exist at z_0

$$\begin{aligned} \text{If } f(z_0) &= 0, \\ f'(z_0) &= 0 \\ &\vdots \\ f^{(m-1)}(z_0) &= 0 \\ f^{(m)}(z_0) &\neq 0 \end{aligned}$$

Then f is said to have a zero of order m at z_0 .

Lemma: $f(z) = (z - z_0)^m \underbrace{g(z)}_{\substack{\uparrow \\ \text{analytic and non-zero at } z_0}}$

Example. $f(z) = z(e^z - 1)$

$$= z^2 \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots \right)$$

has a zero of order $m=2$ at $z_0=0$

$$g(z) = \begin{cases} (e^z - 1)/z & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases} \text{ is analytic at } z=0.$$

Thm. Functions p and q are analytic at z_0 , and $p(z_0) \neq 0$.

If q has a zero of order m at z_0 , then

$\frac{p(z)}{q(z)}$ has a pole of order m there.

$$q(z) = (z - z_0)^m g(z)$$

analytic and non zero

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z - z_0)^m}$$

Example. $f(z) = \frac{1}{z(e^z - 1)}$ has a pole of order 2 at $z_0 = 0$

Corollary: Let two functions p and q be analytic at a point z_0 .

If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$

then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Pf:

$q(z) = (z - z_0) g(z)$, $g(z)$ is analytic and non zero at z_0

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{z - z_0}$$

From Theorem in sec 56, $\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)}$

$$\text{But } g(z_0) = q'(z_0) = \frac{p(z_0)}{q'(z_0)}$$

Example.

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

$$p(z) = \cos z, \quad q(z) = \sin z \quad \text{both entire}$$

The singularities of $f(z)$ occur at zeros of q , or

$$z = n\pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

Since $p(n\pi) = (-1)^n \neq 0$, $q(n\pi) = 0$, and $q'(n\pi) = (-1)^n \neq 0$

each singular point $z = n\pi$ of f is a simple pole,

$$\text{with residue } B_n = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1$$

try $\tan z$