* Taylor's and Laurent's Series

*Taylor Series

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C dz\, \frac{f(z)}{(z-z_0)^{n+1}}$$

$$f(z_0) = \frac{1}{2\pi i}\oint_C dz\, \frac{f(z)}{z-z_0}$$

$$f(z) = \frac{1}{2\pi i}\oint_C dz'\, \frac{f(z')}{z'-z}$$

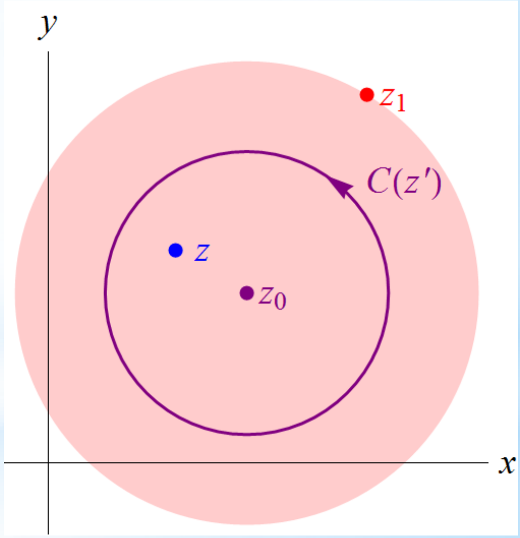
$$\frac{1}{z'-z} = \frac{1}{z'-z_0-(z-z_0)} = \frac{1}{z'-z_0}\left(1-\frac{z-z_0}{z'-z_0}\right)^{-1} = \frac{1}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C dz'\, \frac{f(z')}{(z'-z)^{n+1}}$$

$$\rightarrow \quad f(z) = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!} f^{(n)}(z_0)$$

Taylor series

(f analytic in $R \supset C$)



Let z_1 be the closest singularity from z_0 , then the radius of convergence is $|z_1 - z_0|$.

i.e., series converges for $|z-z_0| < |z_1 - z_0|$

*Laurent Series

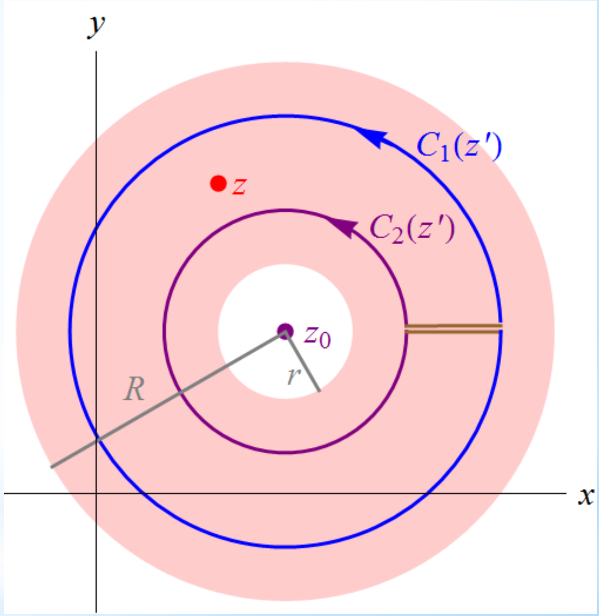
$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^{n+1}}$$

$$f(z_0)=\frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{z-z_0}$$

Let f be analytic within an annular region

$$r\le |z-z_0|\le R$$

$$\rightarrow\qquad f(z)=\frac{1}{2\pi i}\left[\oint_{C_1}-\oint_{C_2}\right]dz'\,\frac{f(z')}{z'-z}$$



C_1 :
$$\frac{1}{z'-z}=\frac{1}{z'-z_0-(z-z_0)}=\frac{1}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

C_2 :
$$\frac{1}{z'-z}=-\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n$$

$\rightarrow$

$$f(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}dz'\,\frac{f(z')}{(z'-z_0)^{n+1}}+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{C_2}dz'\,(z'-z_0)^n f(z')$$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} dz' \frac{f(z')}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} dz' (z' - z_0)^n f(z')$$

$$\sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} dz' (z' - z_0)^n f(z') = \sum_{n=-\infty}^{-1} (z - z_0)^n \oint_{C_2} dz' \frac{f(z')}{(z' - z_0)^{n+1}}$$

→

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Laurent series

$$a_n = \frac{1}{2\pi i} \oint_{C} dz' \frac{f(z')}{(z' - z_0)^{n+1}}$$

C within *f* 's region of analyticity

* Laurent Expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z' - z_0)^{n+1}}$$

$$f(z) = \frac{1}{z(z-1)}$$

Consider expansion about $z_0 = 0$ $\rightarrow$ f is analytic for $0 < |z| < 1$

Expansion via binomial theorem :

$$f(z) = -\left(\frac{1}{z} + \frac{1}{1-z} \right) = -\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n \right)$$

Laurent series :

$$a_n = \frac{1}{2\pi i} \oint_C dz \frac{1}{z^{n+1} z(z-1)} = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_C dz \frac{z^k}{z^{n+2}} = \begin{cases} -1 & n \geq -1 \\ 0 & otherwise \end{cases}$$

$$\rightarrow \qquad f(z) = -\sum_{n=-1}^{\infty} z^n$$