

*Solution of pde's using integral
transforms*

Introduction

- Integral transforms are used to:
- Simplify solutions by eliminating or reducing order of pde in a particular variable
- Offer physical insight into the problem
- Type of transform depends on BC of problem

Introduction

General form for integral transform

$$F(s) = \int_A^B K(s, t) f(t) dt$$

$K(s, t)$ is the kernel of the transform
which decides the type of transform.
 $f(t)$ is transformed to $F(s)$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx) f(x) dx$$

Fourier transform (FT)
and back transform

$$F^{-1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(+ikx) F(k) dk$$

FT suitable for BC where the dependent variable vanishes at ∞
Example: Find the FT of $\exp[-ax^2]$ *Hint:* Complete the square

Fourier transforms of partial derivatives

- pde's are transformed when integral transforms are applied to the pde and BC

$$F[u_x(x, t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_x(x, t) \exp(-ikx) dx = ik F[u(x, t)]$$

$$F[u_{xx}(x, t)] = -k^2 F[u(x, t)]$$

$$F[u_t(x, t)] = \frac{\partial}{\partial t} F[u(x, t)]$$

$$F[u_{tt}(x, t)] = \frac{\partial^2}{\partial t^2} F[u(x, t)]$$

- The first two results are obtained by integration by parts

Fourier Transform of the wave equation wrt x

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{pde}$$

$$u(x,t) \rightarrow 0 \text{ as } x \rightarrow \infty \quad \text{BC}$$

$$u(x,0) = \exp[-ax^2] \quad \text{IC}$$

$$F[u_{tt}] - c^2 F[u_{xx}] = 0 \quad \text{FT pde}$$

$$U''(k,t) + (ck)^2 U(k,t) = 0 \quad \text{ode in t}$$

$$U(k,0) = F[\exp(-ax^2)] = \frac{1}{2\sqrt{a}} \exp(-k^2/4a) \quad \text{FT IC}$$

$$U(k,t) = U(k,0) \exp[-i\omega t] \quad \omega = ck \quad \text{ode solution}$$

$$u(x,t) = \int_{-\infty}^{\infty} U(k,0) \exp[i(kx - \omega t)] dk \quad \text{pde solution}$$

Sine and cosine transforms

Definitions

- The choice of integral transform depends on BC. If we wish to solve a pde with boundaries at $x = 0$ and $x \rightarrow \infty$ then sine and cosine transforms are appropriate.

$$F_S[u] = \frac{2}{\pi} \int_0^{\infty} \sin(kx) u(x, t) dx$$

$$F_S^{-1}[u] = u(x, t) = \int_0^{\infty} \sin(kx) F_S[u] dk$$

$$F_C[u] = \frac{2}{\pi} \int_0^{\infty} \cos(kx) u(x, t) dx$$

$$F_C^{-1}[u] = u(x, t) = \int_0^{\infty} \cos(kx) F_C[u] dk$$

Transforms of partial derivatives

- These results are obtained using integration by parts to eliminate derivatives from the integrand

$$F_S[u_x] = -k F_C[u]$$

$$F_S[u_{xx}] = -k^2 F_S[u] + \frac{2}{\pi} k u(0, t)$$

$$F_C[u_x] = +k F_S[u] - \frac{2}{\pi} u(0, t)$$

$$F_C[u_{xx}] = -k^2 F_C[u] - \frac{2}{\pi} u_x(0, t)$$

Convolution Theorem

- Convolution (or *resultant* or *Faltung*) of f and g ($f * g$) is defined to be

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t - \tau) g(\tau) d\tau$$

- If a function u may be written

$$u = F[f] F[g]$$

$$F^{-1}[u] = F^{-1}\{F[f]F[g]\} = f * g$$

Its inverse Fourier transform is the convolution of f and g .

- Used for solving pde's when we obtain a solution to a particular problem in terms of products of Fourier transforms

Convolution Theorem

Example

$$f(x) = x \quad g(x) = e^{-x^2}$$

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \xi) e^{-\xi^2} d\xi$$

$$\int_{-\infty}^{\infty} \xi e^{-\xi^2} d\xi = 0 \text{ by symmetry}$$

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$$

$$\text{Hence } f * g = \frac{x \sqrt{\pi}}{\sqrt{2\pi}} = \frac{x}{\sqrt{2}}$$

Application of Fourier, sine and cosine transforms

$$u_{tt} = u_{xx} + \sin(\pi x)$$

$$0 < x < 1$$

$$0 < t < \infty$$

$$u(0,t) = 0$$

$$0 < t < \infty$$

$$u(1,t) = 0$$

$$u(x,0) = 1$$

$$0 < x < 1$$

$$u_t(x,0) = 1$$

- We require a finite integral transform for which BC are specified as $u(t)$ (not $u_x(t)$, *etc.*)

Application of Fourier, sine and cosine transforms

Finite sine and cosine transforms

$$S_n(t) = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) u(x, t) dx$$

$$u(x, t) = \sum_{n=1}^{\infty} S_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$C_n(t) = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) u(x, t) dx$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

- Note that the range of integration is $[0, L]$ (finite)

Application of Fourier, sine and cosine transforms

Transforms of partial derivatives

$$S_n[u_x] = -\frac{n\pi}{L} C_n(t)$$

$$S_n[u_{xx}] = -\left(\frac{n\pi}{L}\right)^2 S_n(t) + \frac{2n\pi}{L^2} [u(0,t) + (-1)^{n+1} u(L,t)]$$

$$C_n[u_x] = \frac{n\pi}{L} S_n(t) + \frac{2}{L} [-u(0,t) + (-1)^n u(L,t)]$$

$$C_n[u_{xx}] = -\left(\frac{n\pi}{L}\right)^2 C_n(t) - \frac{2}{L} [u_x(0,t) + (-1)^{n+1} u_x(L,t)]$$

- Note that BC enter as $u(t)$ in $S[u_{xx}]$ and as $u_x(t)$ in $C[u_{xx}]$. The specified BC determine which transform to choose.

Application of Fourier, sine and cosine transforms

Solving inhomogeneous BVP using finite sine transform

- BC involve $u(t)$ rather than $u_x(t)$ so *finite sine* transform

1. Transform pde
2. Transform IC
3. Solve resulting ode
4. Back transform

1. Transform pde

$$S[u_{tt}] = S[u_{xx}] + S[\sin(\pi x)]$$

$$\begin{aligned} S_n''(t) &= -(n\pi)^2 S_n(t) + 2n\pi[u(0,t) + (-1)^{n+1}u(1,t)] + S[\sin(\pi x)] \\ &= -(n\pi)^2 S_n(t) + S[\sin(\pi x)] \end{aligned}$$

$$\begin{aligned} S[\sin(\pi x)] &= 1 & n = 1 \\ &= 0 & n = 2, 3, 4, \dots \end{aligned}$$

Application of Fourier, sine and cosine transforms

Solving inhomogeneous BVP using finite sine transform

2. Transform IC

$$\begin{aligned} S[u(x,0)] = S[1] &= \frac{2}{1} \int_0^L \sin\left(\frac{n\pi x}{1}\right) \cdot 1 dx \\ &= \frac{4}{n\pi} \quad n = 1, 3, 5, \dots \\ &= 0 \quad n = 2, 4, 6, \dots \end{aligned}$$

$$S[u_t(x,0)] = S[0] = 0 \quad n = 1, 2, 3, \dots$$

Application of Fourier, sine and cosine transforms

Solving inhomogeneous BVP using finite sine transform

3. Solve resulting ode

$$\begin{aligned} S_n''(t) + (n\pi)^2 S_n(t) &= 1 & n = 1 & \text{ode} \\ &= 0 & n = 2, 3, 4, \dots & \end{aligned}$$

$$S_n(0) = \frac{4}{n\pi} \quad n = 1, 3, 5, \dots \quad \text{IC}$$

$$= 0 \quad n = 2, 4, 6, \dots$$

$$S_n'(t) = 0 \quad n = 1, 2, 3, \dots$$

Application of Fourier, sine and cosine transforms

Solving inhomogeneous BVP using finite sine transform

$$S_1(t) = A \cos(\pi t) + \pi^{-2}$$

$$S_n(t) = 0 \quad n = 2, 4, 6, \dots \quad \text{solution}$$

$$S_n(t) = 4 (n\pi)^{-1} \cos(n\pi t) \quad n = 3, 5, 7, \dots$$

$$S_1(0) = A + \pi^{-2} = 4\pi^{-1} \quad A = (4\pi - 1)\pi^{-2}$$

$$S_n(0) = 0 \quad n = 2, 4, 6, \dots \quad \text{IC satisfied}$$

$$S_n(0) = 4 (n\pi)^{-1} \quad n = 3, 5, 7, \dots$$

$$S_1'(0) = -A\pi \sin(\pi \cdot 0) = 0$$

$$S_n'(0) = 0 \quad n = 2, 4, 6, \dots \quad \text{IC satisfied}$$

$$S_n'(0) = -4 \sin(n\pi \cdot 0) = 0 \quad n = 3, 5, 7, \dots$$

ode satisfied

$$\begin{aligned} S_n''(t) + (n\pi)^2 S_n(t) &= -A \pi^2 \cos(\pi t) + A \pi^2 \cos(\pi t) + 1 \quad n = 1 \\ &= -4n\pi \cos(n\pi t) + 4n\pi \cos(n\pi t) \quad n = 3, 5, \end{aligned}$$

Application of Fourier, sine and cosine transforms

Solving inhomogeneous BVP using finite sine transform

4. Back transform

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} S_n(t) \sin\left(\frac{n\pi x}{1}\right) \\&= \left[A \cos(\pi t) + \pi^{-2} \right] \sin(\pi x) + \\&\quad \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n+1} \cos[(2n+1)\pi t] \sin[(2n+1)\pi x]\end{aligned}$$

Summary of Integral Transforms

Kernel	Boundary Conditions	Restrictions
$\exp(ikx)$	$u(x, t) \rightarrow 0$ as $x \rightarrow \pm\infty$	No FT exists for many functions
$\sin(kx)$	$u(0, t) = f(t)$ $u(x, t) \rightarrow 0$ as $x \rightarrow \pm\infty$	PDE must have no 1 st order derivatives wrt x
$\cos(kx)$	$u_x(0, t) = f(t)$ $u(x, t) \rightarrow 0$ as $x \rightarrow \pm\infty$	PDE must have no 1 st order derivatives wrt x
finite sine	$u(0, t), u(L, t)$	no mixed BC
finite cosine	$u_x(0, t), u_x(L, t)$	no mixed BC
$\exp(-st)$	$u(0, t), u_x(0, t)$ $u(x, t) \rightarrow 0$ as $x \rightarrow \pm\infty$ BC can be mixed	$u(x, t)$ does not grow faster than exponentially for $t > T$