The z-Transform

and its Applications

z-Transform

- The z-transform is the most general concept for the transformation of discrete-time series.
- The Laplace transform is the more general concept for the transformation of continuous time processes.
- For example, the Laplace transform allows you to transform a differential equation, and its corresponding initial and boundary value problems, into a space in which the equation can be solved by ordinary algebra.
- The switching of spaces to transform calculus problems into algebraic operations on transforms is called operational calculus. The Laplace and z transforms are the most important methods for this purpose.

The Transforms

The Laplace transform of a function *f*(*t*):

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

The one-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

The two-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Relationship to Fourier Transform

Note that expressing the complex variable *z* in polar form reveals the relationship to the Fourier transform:

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n}, \; or$$

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-i\omega n}, \; and \; if \; r = 1,$$

$$X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n}$$

which is the Fourier transform of $x(n)$.

Region of Convergence

The z-transform of *x*(*n*) can be viewed as the Fourier transform of *x*(*n*) multiplied by an exponential sequence r^{-n}, and the z-transform may converge even when the Fourier transform does not.

By redefining convergence, it is possible that the Fourier transform may converge when the z-transform does not.

For the Fourier transform to converge, the sequence must have finite energy, or:

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

Convergence, continued

The power series for the z-transform is called a Laurent series:

$$X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence, and therefore the z-transform and all its derivatives must be continuous functions of *z* inside the region of convergence.

In general, the Laurent series will converge in an annular region of the z-plane.

Some Special Functions

First we introduce the **Dirac delta function** **(or unit sample function):**

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \quad \textbf{or} \quad \delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

This allows an arbitrary sequence *x*(*n*) or continuous-time function *f*(*t*) to be expressed as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$f(t) = \int_{-\infty}^{\infty} f(x)\delta(x-t)dt$$

Convolution, Unit Step

These are referred to as discrete-time or continuous-time **convolution, and are denoted by:**

$$x(n) = x(n) * \delta(n)$$

$$f(t) = f(t) * \delta(t)$$

We also introduce the unit step function:

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{or} \quad u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note also:

$$u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

Poles and Zeros

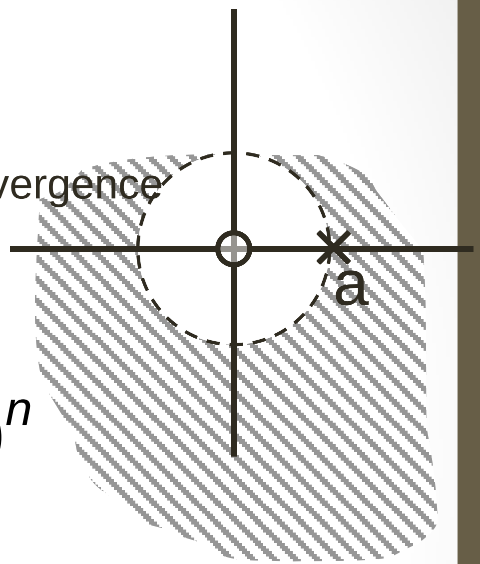
When $X(z)$ is a rational function, i.e., a ration of polynomials in z, then:

1. **The roots of the numerator polynomial are referred to as the zeros of $X(z)$, and**
2. **The roots of the denominator polynomial are referred to as the poles of $X(z)$.**

Note that no poles of $X(z)$ can occur within the region of convergence since the z-transform does not converge at a pole.

Furthermore, the region of convergence is bounded by poles.

Example



$$x(n) = a^n u(n)$$

The z-transform is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Which converges to:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad for \quad |z| > |a|$$

Clearly, *X*(*z*) has a zero at *z* = 0 and a pole at *z* = *a*.

Convergence of Finite Sequences

Suppose that only a finite number of sequence values are nonzero, so that:

$$X(z) = \sum_{n=n_1}^{n_2} x(n) z^{-n}$$

Where n_1 and n_2 are finite integers. Convergence requires

$$|x(n)| < \infty \ \ for \ \ n_1 \le n \le n_2.$$

So that finite-length sequences have a region of convergence that is at least $0 < |z| < \infty$, and may include either $z = 0$ or $z = \infty$.

Inverse z-Transform

The inverse z-transform can be derived by using Cauchy's integral theorem. Start with the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Multiply both sides by z^{k-1} and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely within the region of convergence of $X(z)$:

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = \frac{1}{2\pi i}\oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi i}\oint_C z^{-n+k-1} dz$$

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = x(n) \text{ is the inverse z-transform.}$$

Properties

- z-transforms are linear:

$$\mathbf{Z}\left[ax(n) + by(n)\right] = aX(z) + bY(z)$$

- The transform of a shifted sequence:

$$\mathbf{Z}\left[x(n + n_0)\right] = z^{n_0} X(z)$$

- Multiplication:

$$\mathbf{Z}\left[a^n x(n)\right] = Z(a^{-1}z)$$

But multiplication will affect the region of convergence and all the pole-zero locations will be scaled by a factor of a.

Convolution of Sequences

$$w(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

Then

$$W(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)y(n-k) \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k) z^{-n}$$

let $m = n - k$

$$W(z) = \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{m=-\infty}^{\infty} y(m) z^{-m} \right] z^{-k}$$

$W(z) = X(z)Y(z)$ for values of z inside the regions of convergenc e of both.