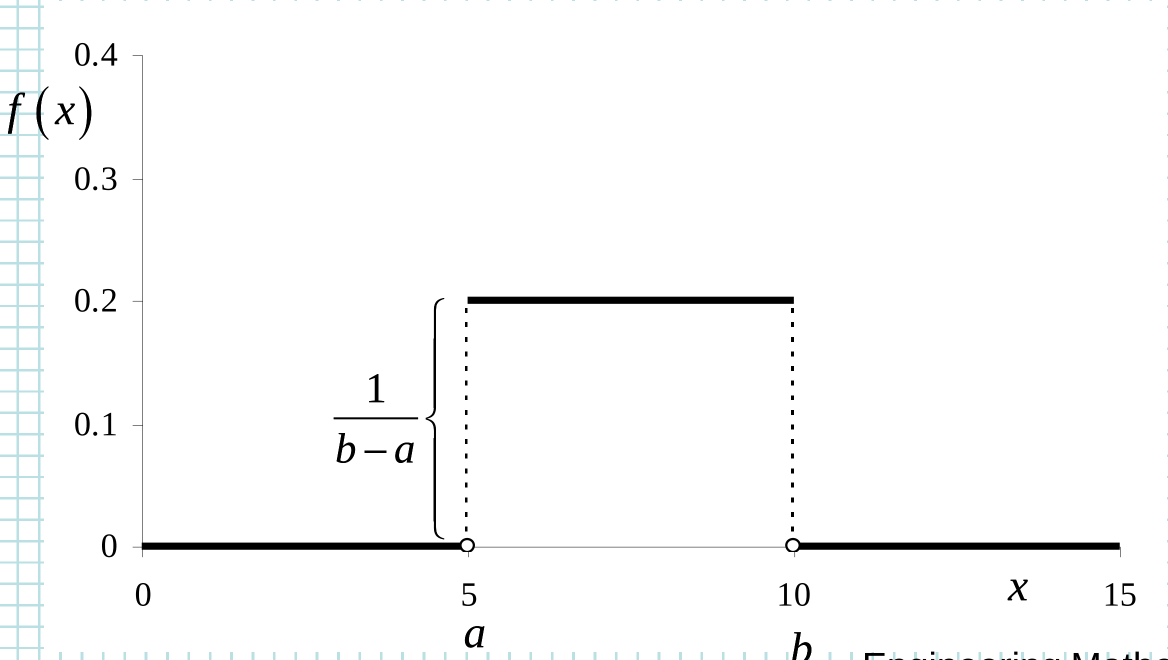
MOMENT GENERATING FUNCTIONS

Continuous Distributions

The Uniform distribution from *a* to *b*

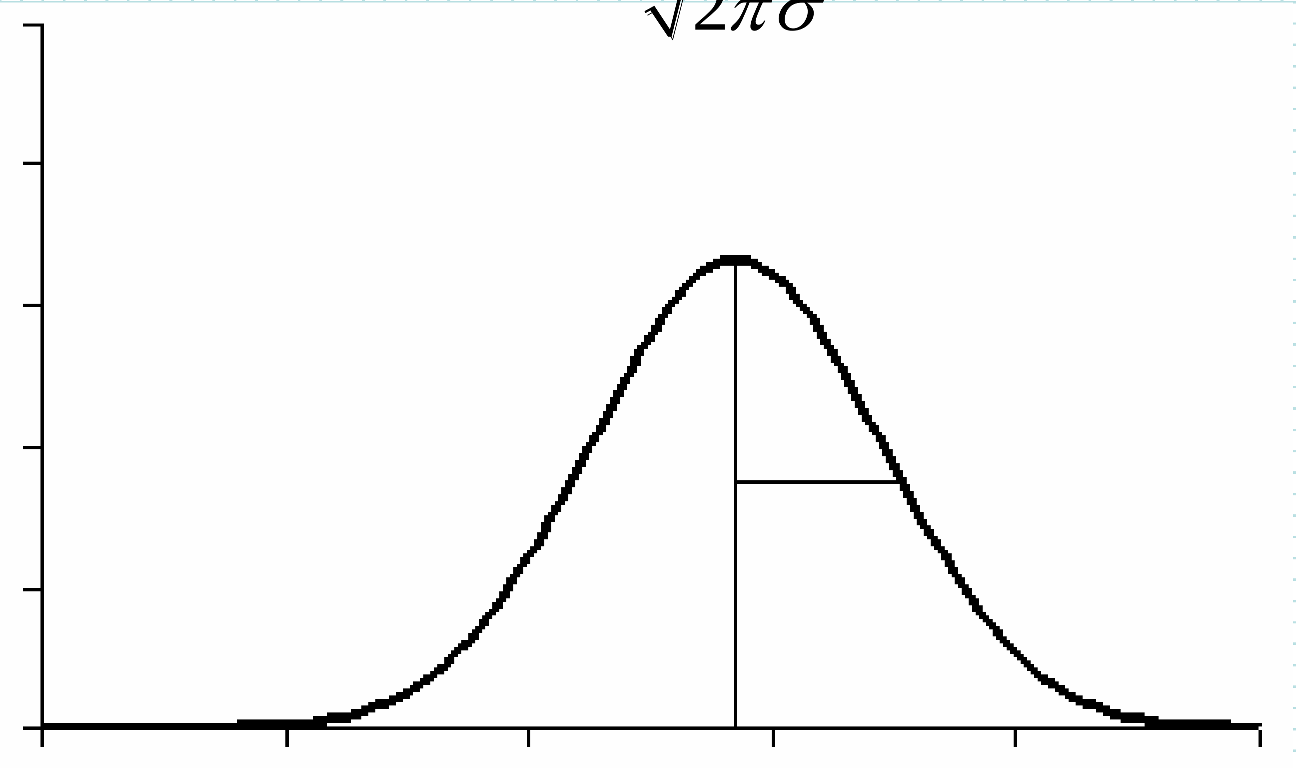
$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



The Normal distribution

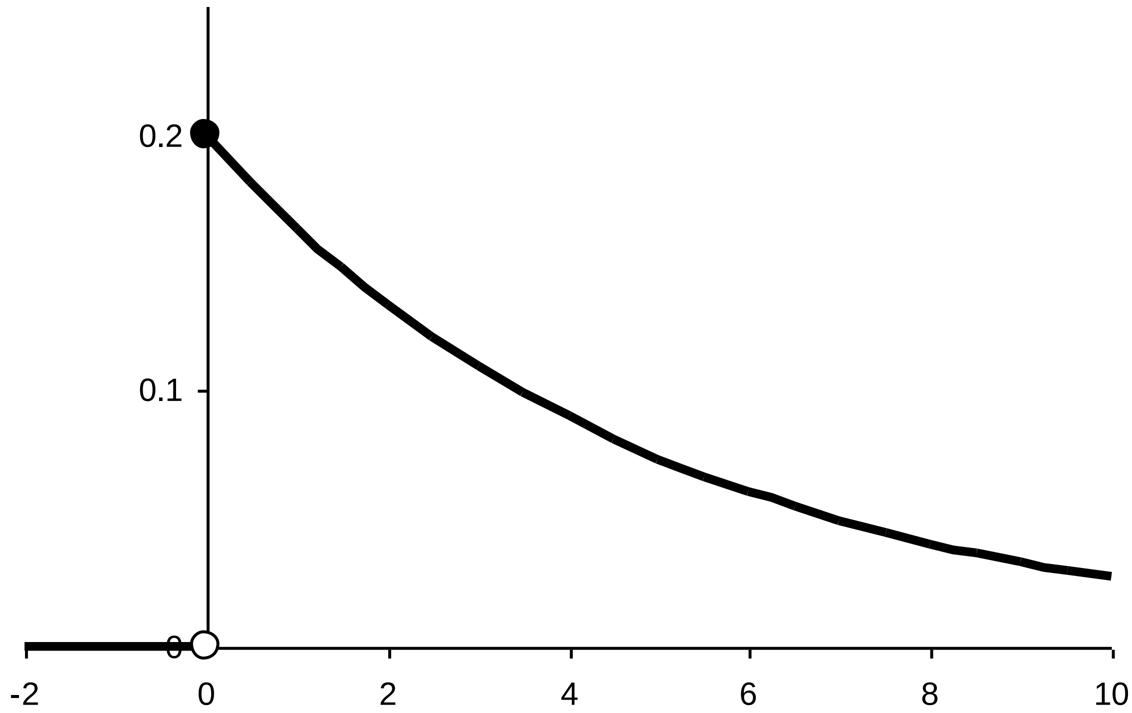
(mean μ, standard deviation σ)

$$f\left(x\right)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{\left(x-\mu\right)^2}{2\sigma^2}}$$



The Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Weibull distribution with parameters α and β.

$$F(x) = 1 - e^{-\frac{\alpha}{\beta} x^{\beta}}$$

$$f(x) = F'(x) = \alpha x^{\beta-1} e^{-\frac{\alpha}{\beta} x^{\beta}} \qquad x \geq 0$$

The Gamma distribution

Let the continuous random variable X have density function:

$$f(x) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Then X is said to have a **Gamma distribution** with parameters α and λ.

Expectation of functions of Random Variables

X is **discrete**

$$E\left[g(X)\right]=\sum_{x} g(x)\,p(x)=\sum_{i} g(x_i)\,p(x_i)$$

X is **continuous**

$$E\left[g(X)\right]=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx$$

Moment generating functions

Moment Generating function of a R.V. *X*

$$m_X(t) = E\left[e^{tX}\right] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

Examples

1. The Binomial distribution (parameters p, n)

$$m_X(t) = E\left[e^{tX}\right] = \sum_x e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} \left(e^t p\right)^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

$$= (a+b)^n = \left(e^t p + 1 - p\right)^n$$

2. The Poisson distribution (parameter λ)

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0, 1, 2, \ldots$$

The moment generating function of X , $m_X(t)$ is:

$$m_X(t) = E\left[e^{tX}\right] = \sum_x e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!} = e^{-\lambda} e^{\lambda e^t} \quad \text{using} \quad e^u = \sum_{x=0}^{\infty} \frac{u^x}{x!}$$

$$= e^{\lambda\left(e^t - 1\right)}$$

3. The Exponential distribution (parameter λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The moment generating function of X , $m_X(t)$ is:

$$m_X(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \int_{0}^{\infty} \lambda e^{(t-\lambda)x}\,dx = \left[\lambda \frac{e^{(t-\lambda)x}}{t-\lambda}\right]_{0}^{\infty}$$

$$= \begin{cases} \dfrac{\lambda}{\lambda - t} & t < \lambda \\ \text{undefined} & t \geq \lambda \end{cases}$$

4. The Standard Normal distribution ($\mu = 0$, $\sigma = 1$)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The moment generating function of X , $m_X(t)$ is:

$$m_X(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2tx}{2}} dx$$

We will now use the fact that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a}} e^{-\frac{(x-b)^2}{2a^2}} dx = 1 \text{ for all } a > 0, b$$

We have completed the square

$$m_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2tx}{2}} dx = e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2tx+t^2}{2}} dx$$

$$= e^{\frac{t^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx}_{} = e^{\frac{t^2}{2}}$$

This is 1

4. The Gamma distribution (parameters α, λ)

$$f(x) = \begin{cases} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The moment generating function of X , $m_X(t)$ is:

$$m_X(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_{0}^{\infty} e^{tx} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \int_{0}^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx$$

We use the fact

$$\int_0^\infty \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} dx = 1 \text{ for all } a > 0, b > 0$$

$$m_X(t) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \underbrace{\int_0^\infty \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx}_{\text{Equal to 1}} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$$

Equal to 1

Properties of
Moment Generating Functions

1. $m_X(0) = 1$

$$m_X(t) = E\left(e^{tX}\right), \text{ hence } m_X(0) = E\left(e^{0\cdot X}\right) = E(1) = 1$$

Note: the moment generating functions of the following distributions satisfy the property $m_X(0) = 1$

i) Binomial Dist'n $m_X(t) = \left(e^t p + 1 - p\right)^n$

ii) Poisson Dist'n $m_X(t) = e^{\lambda\left(e^t - 1\right)}$

iii) Exponential Dist'n $m_X(t) = \left(\dfrac{\lambda}{\lambda - t}\right)$

iv) Std Normal Dist'n $m_X(t) = e^{\frac{t^2}{2}}$

v) Gamma Dist'n $m_X(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^{\alpha}$

2. $$m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \frac{\mu_3}{3!}t^3 + \ldots + \frac{\mu_k}{k!}t^k + \ldots$$

We use the expansion of the exponential function:

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots + \frac{u^k}{k!} + \ldots$$

$$m_X(t) = E\left(e^{tX}\right)$$

$$= E\left(1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \ldots + \frac{t^k}{k!}X^k + \ldots\right)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E\left(X^2\right) + \frac{t^3}{3!}E\left(X^3\right) + \ldots + \frac{t^k}{k!}E\left(X^k\right) + \ldots$$

$$= 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \ldots + \frac{t^k}{k!}\mu_k + \ldots$$

3. $$m_X^{(k)}(0)=\frac{d^k}{dt^k}m_X(t)\bigg|_{t=0}=\mu_k$$

Now

$$m_X(t)=1+\mu_1 t+\frac{\mu_2}{2!}t^2+\frac{\mu_3}{3!}t^3+\ldots+\frac{\mu_k}{k!}t^k+\ldots$$

$$m_X'(t)=\mu_1+\frac{\mu_2}{2!}2t+\frac{\mu_3}{3!}3t^2+\ldots+\frac{\mu_k}{k!}kt^{k-1}+\ldots$$

$$=\mu_1+\mu_2 t+\frac{\mu_3}{2!}t^2+\ldots+\frac{\mu_k}{(k-1)!}t^{k-1}+\ldots$$

and $m_X'(0)=\mu_1$

$$m_X''(t)=\mu_2+\mu_3 t+\frac{\mu_4}{2!}t+\ldots+\frac{\mu_k}{(k-2)!}t^{k-2}+\ldots$$

and $m_X''(0)=\mu_2$

continuing we find $m_X^{(k)}(0)=\mu_k$

Property 3 is very useful in determining the moments of a random variable X.

Examples

i) Binomial Dist'n $m_X(t) = \left(e^t p + 1 - p\right)^n$

$$m'_X(t) = n\left(e^t p + 1 - p\right)^{n-1}\left(pe^t\right)$$

$$m'_X(0) = n\left(e^0 p + 1 - p\right)^{n-1}\left(pe^0\right) = np = \mu_1 = \mu$$

$$m''_X(t) = np\left[(n-1)\left(e^t p + 1 - p\right)^{n-2}\left(e^t p\right)e^t + \left(e^t p + 1 - p\right)^{n-1} e^t\right]$$

$$= npe^t\left(e^t p + 1 - p\right)^{n-2}\left[(n-1)\left(e^t p\right) + \left(e^t p + 1 - p\right)\right]$$

$$= npe^t\left(e^t p + 1 - p\right)^{n-2}\left[ne^t p + 1 - p\right]$$

$$= np\left[np + 1 - p\right] = np\left[np + q\right] = n^2 p^2 + npq = \mu_2$$

ii) Poisson Dist'n $m_X(t) = e^{\lambda(e^t - 1)}$

$$m'_X(t) = e^{\lambda(e^t-1)}\left[\lambda e^t\right] = \lambda e^{\lambda(e^t-1)+t}$$

$$m''_X(t) = \lambda e^{\lambda(e^t-1)+t}\left[\lambda e^t + 1\right] = \lambda^2 e^{\lambda(e^t-1)+2t} + \lambda e^{\lambda(e^t-1)+t}$$

$$m'''_X(t) = \lambda^2 e^{\lambda(e^t-1)+2t}\left[\lambda e^t + 2\right] + \lambda e^{\lambda(e^t-1)+t}\left[\lambda e^t + 1\right]$$

$$= \lambda^2 e^{\lambda(e^t-1)+2t}\left[\lambda e^t + 3\right] + \lambda e^{\lambda(e^t-1)+t}$$

$$= \lambda^3 e^{\lambda(e^t-1)+3t} + 3\lambda^2 e^{\lambda(e^t-1)+2t} + \lambda e^{\lambda(e^t-1)+t}$$

To find the moments we set t = 0.

$$\mu_1 = m_X'(0) = \lambda e^{\lambda\left(e^0 - 1\right) + 0} = \lambda$$

$$\mu_2 = m_X''(0) = \lambda^2 e^{\lambda\left(e^0 - 1\right) + 0} + \lambda e^{\lambda\left(e^0 - 1\right) + 0} = \lambda^2 + \lambda$$

$$\mu_3 = m_X'''(0) = \lambda^3 e^0 + 3\lambda^2 e^{0t} + \lambda e^0 = \lambda^3 + 3\lambda^2 + \lambda$$

iii) Exponential Dist'n $m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)$

$$m'_X(t) = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right) = \lambda \frac{d(\lambda - t)^{-1}}{dt}$$

$$= \lambda(-1)(\lambda - t)^{-2}(-1) = \lambda(\lambda - t)^{-2}$$

$$m''_X(t) = \lambda(-2)(\lambda - t)^{-3}(-1) = 2\lambda(\lambda - t)^{-3}$$

$$m'''_X(t) = 2\lambda(-3)(\lambda - t)^{-4}(-1) = 2(3)\lambda(\lambda - t)^{-4}$$

$$m_X^{(4)}(t) = 2(3)\lambda(-4)(\lambda - t)^{-5}(-1) = (4!)\lambda(\lambda - t)^{-5}$$

$$\vdots$$

$$m_X^{(k)}(t) = (k!)\lambda(\lambda - t)^{-k-1}$$

Thus

$$\mu_1 = \mu = m_X'(0) = \lambda(\lambda)^{-2} = \frac{1}{\lambda}$$

$$\mu_2 = m_X''(0) = 2\lambda(\lambda)^{-3} = \frac{2}{\lambda^2}$$

$$\vdots$$

$$\mu_k = m_X^{(k)}(0) = (k!)\lambda(\lambda)^{-k-1} = \frac{k!}{\lambda^k}$$

The moments for the exponential distribution can be calculated in an alternative way. This is note by expanding $m_X(t)$ in powers of t and equating the coefficients of t^k to the coefficients in:

$$m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \frac{\mu_3}{3!}t^3 + \ldots + \frac{\mu_k}{k!}t^k + \ldots$$

$$m_X(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1 - t/\lambda} = \frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \cdots$$

Equating the coefficients of t^k we get:

$$\frac{\mu_k}{k!} = \frac{1}{\lambda^k} \text{ or } \mu_k = \frac{k!}{\lambda^k}$$

The moments for the standard normal distribution

$$m_X(t) = e^{\frac{t^2}{2}}$$

We use the expansion of e^u.

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^k}{k!} + \cdots$$

$$m_X(t) = e^{\frac{t^2}{2}} = 1 + \left(\tfrac{t^2}{2}\right) + \frac{\left(\frac{t^2}{2}\right)^2}{2!} + \frac{\left(\frac{t^2}{2}\right)^3}{3!} + \cdots + \frac{\left(\frac{t^2}{2}\right)^k}{k!} + \cdots$$

$$= 1 + \tfrac{1}{2}t^2 + \frac{1}{2^2 2!}t^4 + \frac{1}{2^3 3!}t^6 + \cdots + \frac{1}{2^k k!}t^{2k} + \cdots$$

We now equate the coefficients t^k in:

$$m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \ldots + \frac{\mu_k}{k!}t^k + \ldots + \frac{\mu_{2k}}{(2k)!}t^{2k} + \ldots$$

If k is odd: $\mu_k = 0.$

For even $2k$: $$\frac{\mu_{2k}}{(2k)!} = \frac{1}{2^k k!}$$

or $$\mu_{2k} = \frac{(2k)!}{2^k k!}$$

Thus $$\mu_1 = 0, \mu_2 = \frac{2!}{2} = 1, \mu_3 = 0, \mu_4 = \frac{4!}{2^2 (2!)} = 3$$

Summary

Moments

Moment generating functions

Moments of Random Variables

$$\mu_k = E\left(X^k\right)$$

$$= \begin{cases} \sum_x x^k p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^k f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

The moment generating function

$$m_X(t) = E\left[e^{tX}\right] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

Examples

1. The Binomial distribution (parameters *p*, *n*)

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

$$m_X(t) = \left(e^t p + 1 - p\right)^n = \left(e^t p + q\right)^n$$

2. The Poisson distribution (parameter λ)

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0, 1, 2, \ldots$$

$$m_X(t) = e^{\lambda\left(e^t - 1\right)}$$

3. The Exponential distribution (parameter λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$m_X(t) = \begin{cases} \dfrac{\lambda}{\lambda - t} & t < \lambda \\ \text{undefined} & t \geq \lambda \end{cases}$$

4. The Standard Normal distribution ($\mu = 0$, $\sigma = 1$)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$m_X(t) = e^{\frac{t^2}{2}}$$

5. The Gamma distribution (parameters α, λ)

$$f(x) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$m_X(t) = \left(\frac{\lambda}{\lambda - t} \right)^{\alpha}$$

6. The Chi-square distribution (degrees of freedom ν)

($\alpha = \nu/2$, $\lambda = 1/2$)

$$f(x) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{1}{2}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$m_X(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t} \right)^{\frac{\nu}{2}} = (1 - 2t)^{-\frac{\nu}{2}}$$

Properties of Moment Generating Functions

1. $m_X(0) = 1$

2. $$m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \frac{\mu_3}{3!}t^3 + \ldots + \frac{\mu_k}{k!}t^k + \ldots$$

3. $$m_X^{(k)}(0) = \frac{d^k}{dt^k} m_X(t)\bigg|_{t=0} = \mu_k$$

i.e. $m_X'(0) = \mu_1$

$m_X''(0) = \mu_2$

$m_X'''(0) = \mu_3$, etc

The log of Moment Generating Functions

Let $l_X(t) = \ln m_X(t)$ = the log of the moment generating function

Then $l_X(0) = \ln m_X(0) = \ln 1 = 0$

$$l'_X(t) = \frac{1}{m_X(t)} m'_X(t) = \frac{m'_X(t)}{m_X(t)} \qquad l'_X(0) = \frac{m'_X(0)}{m_X(0)} = \mu_1 = \mu$$

$$l''_X(t) = \frac{m''_X(t) m_X(t) - \left[m'_X(t)\right]^2}{\left[m_X(t)\right]^2}$$

$$l''_X(0) = \frac{m''_X(0) m_X(0) - \left[m'_X(0)\right]^2}{\left[m_X(0)\right]^2} = \mu_2 - [\mu_1]^2 = \sigma^2$$

Thus $l_X(t) = \ln m_X(t)$ is very useful for calculating the mean and variance of a random variable

1. $l'_X(0) = \mu$
2. $l''_X(0) = \sigma^2$

Examples

1. The Binomial distribution (parameters p, n)

$$m_X(t) = \left(e^t p + 1 - p\right)^n = \left(e^t p + q\right)^n$$

$$l_X(t) = \ln m_X(t) = n \ln\left(e^t p + q\right)$$

$$l_X'(t) = n \frac{1}{e^t p + q} e^t p$$

$$\mu = l_X'(0) = n \frac{1}{p + q} p = np$$

$$l_X''(t) = n \frac{e^t p\left(e^t p + q\right) - e^t p\left(e^t p\right)}{\left(e^t p + q\right)^2}$$

$$\sigma^2 = l_X''(0) = n \frac{p(p + q) - p(p)}{(p + q)^2} = npq$$

2. The Poisson distribution (parameter λ)

$$m_X(t) = e^{\lambda\left(e^t - 1\right)}$$

$$l_X(t) = \ln m_X(t) = \lambda\left(e^t - 1\right)$$

$$l'_X(t) = \lambda e^t$$

$$l''_X(t) = \lambda e^t$$

$$\mu = l'_X(0) = \lambda$$

$$\sigma^2 = l''_X(0) = \lambda$$

3. The Exponential distribution (parameter λ)

$$m_X(t) = \begin{cases} \dfrac{\lambda}{\lambda - t} & t < \lambda \\ \text{undefined} & t \geq \lambda \end{cases}$$

$$l_X(t) = \ln m_X(t) = \ln \lambda - \ln(\lambda - t) \quad \text{if} \quad t < \lambda$$

$$l'_X(t) = \frac{1}{\lambda - t} = (\lambda - t)^{-1}$$

$$l''_X(t) = -1(\lambda - t)^{-2}(-1) = \frac{1}{(\lambda - t)^2}$$

Thus $\mu = l'_X(0) = \dfrac{1}{\lambda}$ and $\sigma^2 = l''_X(0) = \dfrac{1}{\lambda^2}$

4. The Standard Normal distribution ($\mu = 0$, $\sigma = 1$)

$$m_X(t) = e^{\frac{t^2}{2}}$$

$$l_X(t) = \ln m_X(t) = \tfrac{t^2}{2}$$

$$l_X'(t) = t, \quad l_X''(t) = 1$$

Thus $\mu = l_X'(0) = 0$ and $\sigma^2 = l_X''(0) = 1$

5. The Gamma distribution (parameters α, λ)

$$m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$$

$$l_X(t) = \ln m_X(t) = \alpha\left[\ln\lambda - \ln(\lambda - t)\right]$$

$$l'_X(t) = \alpha\left[\frac{1}{\lambda - t}\right] = \frac{\alpha}{\lambda - t}$$

$$l''_X(t) = \alpha(-1)(\lambda - t)^{-2}(-1) = \frac{\alpha}{(\lambda - t)^2}$$

Hence $\mu = l'_X(0) = \dfrac{\alpha}{\lambda}$ and $\sigma^2 = l''_X(0) = \dfrac{\alpha}{\lambda^2}$

6. The Chi-square distribution (degrees of freedom ν)

$$m_X(t) = (1-2t)^{-\frac{\nu}{2}}$$

$$l_X(t) = \ln m_X(t) = -\frac{\nu}{2}\ln(1-2t)$$

$$l_X'(t) = -\frac{\nu}{2}\frac{1}{1-2t}(-2) = \frac{\nu}{1-2t}$$

$$l_X''(t) = \nu(-1)(1-2t)^{-2}(-2) = \frac{2\nu}{(1-2t)^2}$$

Hence $\mu = l_X'(0) = \nu$ and $\sigma^2 = l_X''(0) = 2\nu$

Summary of Discrete Distributions

Name	probability function p(x)	Mean	Variance	Moment generating function $M_X(t)$
Discrete Uniform	$p(x) = \frac{1}{N} \quad x=1,2,\ldots,N$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\frac{e^t}{N}\frac{e^{tN}-1}{e^t-1}$
Bernoulli	$p(x) = \begin{cases} p & x=1 \\ q & x=0 \end{cases}$	p	pq	$q + pe^t$
Binomial	$p(x) = \binom{N}{x} p^x q^{N-x}$	Np	Npq	$(q + pe^t)^N$
Geometric	$p(x) = pq^{x-1} \quad x=1,2,\ldots$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$
Negative Binomial	$p(x) = \binom{x-1}{k-1} p^k q^{x-k}$ $x=k,k+1,\ldots$	$\frac{k}{p}$	$\frac{kq}{p^2}$	$\left[\frac{pe^t}{1-qe^t}\right]^k$
Poisson	$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x=1,2,\ldots$	λ	λ	$e^{\lambda(e^t-1)}$
Hypergeometric	$p(x) = \frac{\binom{A}{x}\binom{N-A}{n-x}}{\binom{N}{n}}$	$n\left(\frac{A}{N}\right)$	$n\left(\frac{A}{N}\right)\left(1-\frac{A}{N}\right)\left(\frac{N-n}{N-1}\right)$	not useful

Summary of Continuous Distributions

Name	probability density function f(x)	Mean	Variance	Moment generating function $M_X(t)$
Continuous Uniform	$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{[b-a]t}$
Exponential	$f(x) = \begin{cases} le^{-lx} & x = 0 \\ 0 & x < 0 \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left[\frac{\lambda}{\lambda - t}\right]$ for $t < \lambda$
Gamma	$f(x) = f(x) = \begin{cases} \frac{l^a}{G(a)} x^{a-1} e^{-lx} & x = 0 \\ 0 & x < 0 \end{cases}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left[\frac{\lambda}{\lambda - t}\right]^{\alpha}$ for $t < \lambda$
χ^2 ν d.f.	$f(x) = \begin{cases} \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)} x^{\nu/2-1} e^{-(1/2)x} & x \ ? \ 0 \\ 0 & x < 0 \end{cases}$	ν	2ν	$\left[\frac{1}{1-2t}\right]^{\nu/2}$ for $t < 1/2$
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2	$e^{t\mu+(1/2)t^2\sigma^2}$
Weibull	$f(x) = \begin{cases} \frac{\gamma}{\theta} x^{\gamma-1} e^{-x^{\gamma}/\theta} & x \ ? \ 0 \\ 0 & x < 0 \end{cases}$	$\theta^{2/\gamma}\Gamma\left(\frac{\gamma+1}{\gamma}\right)$	$\theta^{2/\gamma}\left\{\Gamma\left(\frac{\gamma+2}{\gamma}\right)-\left[\Gamma\left(\frac{\gamma+1}{\gamma}\right)\right]^2\right\}$	not avail.