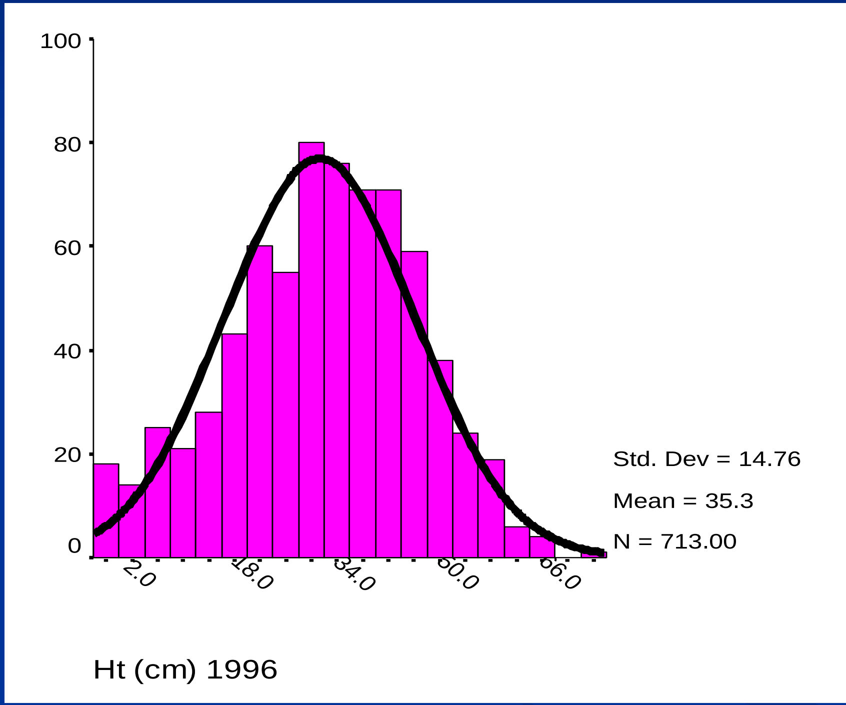
# The Binomial, Poisson, and Normal Distributions

# Probability distributions

- We use probability distributions because they work –they fit lots of data in real world




Height (cm) of *Hypericum cumulicola* at Archbold Biological Station

# Random variable

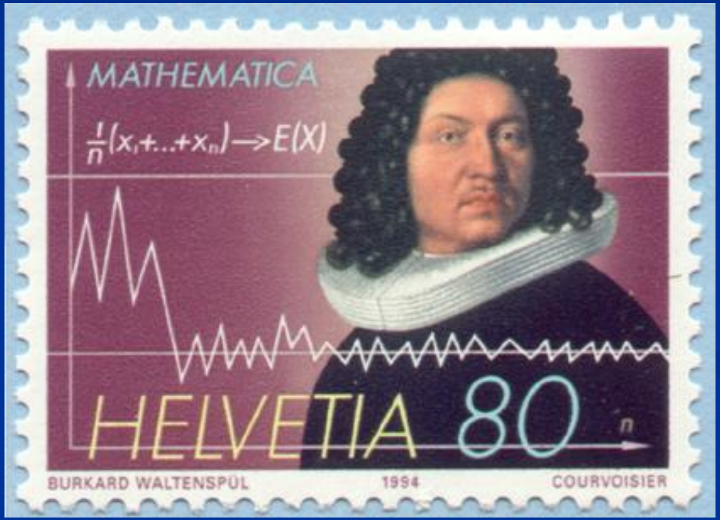
- The mathematical rule (or function) that assigns a given numerical value to each possible outcome of an experiment in the sample space of interest.

# Types of Random variables

- Discrete random variables
- Continuous random variables

# The Binomial Distribution
## Bernoulli Random Variables

- Imagine a simple trial with only two possible outcomes
  - Success (*S*)
  - Failure (*F*)




Jacob Bernoulli (1654-1705)

- Examples
  - Toss of a coin (heads or tails)
  - Sex of a newborn (male or female)
  - Survival of an organism in a region (live or die)

# The Binomial Distribution
## Overview

- Suppose that the probability of success is $p$

- What is the probability of failure?
  - $q = 1 - p$

- Examples
  - Toss of a coin ($S$ = head): $p = 0.5 \Rightarrow q = 0.5$
  - Roll of a die ($S$ = 1): $p = 0.1667 \Rightarrow q = 0.8333$
  - Fertility of a chicken egg ($S$ = fertile): $p = 0.8 \Rightarrow q = 0.2$

# The Binomial Distribution
## Overview

- Imagine that a trial is repeated $n$ times
- Examples
  - A coin is tossed 5 times
  - A die is rolled 25 times
  - 50 chicken eggs are examined
- Assume $p$ remains constant from trial to trial and that the trials are statistically independent of each other

# The Binomial Distribution
## Overview

- What is the probability of obtaining $x$ successes in $n$ trials?

- Example
  - What is the probability of obtaining 2 heads from a coin that was tossed 5 times?

$$P(HHTTT) = (1/2)^5 = 1/32$$

# The Binomial Distribution
## Overview

- But there are more possibilities:

| | | | |
|---|---|---|---|
| HHTTT | HTHTT | HTTHT | HTTTH |
| | THHTT | THTHT | THTTH |
| | | TTHHT | TTHTH |
| | | | TTTHH |

$P(2 \text{ heads}) = 10 \times 1/32 = 10/32$

# The Binomial Distribution
## Overview

- In general, if trials result in a series of success and failures,

$$FFSFFFFSFSFSSFFFFFSF\ldots$$

Then the probability of $x$ successes <u>in that order</u> is

$$\begin{aligned} P(x) &= q \cdot q \cdot p \cdot q \cdot \ldots \\ &= p^x \cdot q^{n-x} \end{aligned}$$

# The Binomial Distribution
## Overview

- However, if order is not important, then

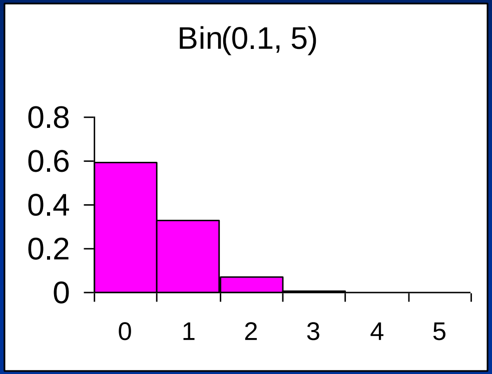
$$P(x) = \frac{n!}{x!(n-x)!} p^x \cdot q^{n-x}$$

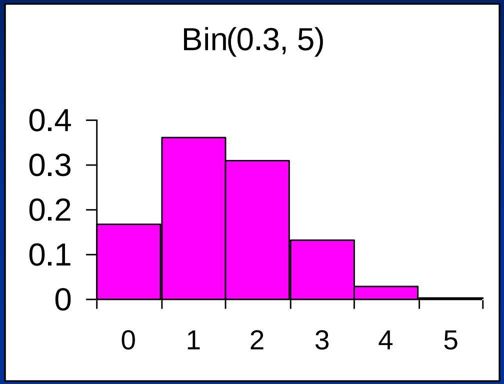
where $\frac{n!}{x!(n-x)!}$ is the number of ways to obtain $x$ successes

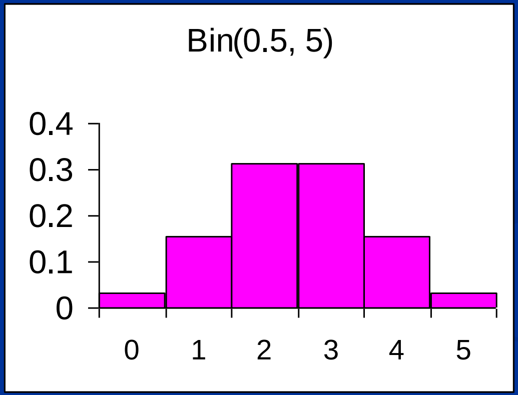
in $n$ trials, and $i! = i \cdot (i-1) \cdot (i-2) \cdot \ldots \cdot 2 \cdot 1$

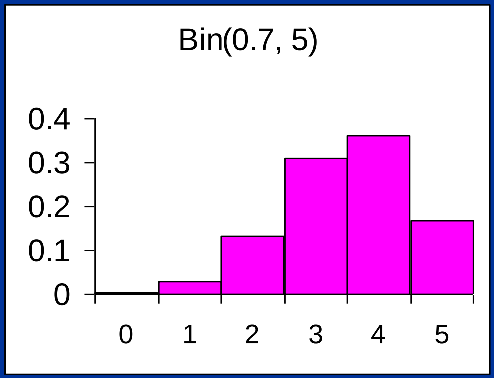
# The Binomial Distribution

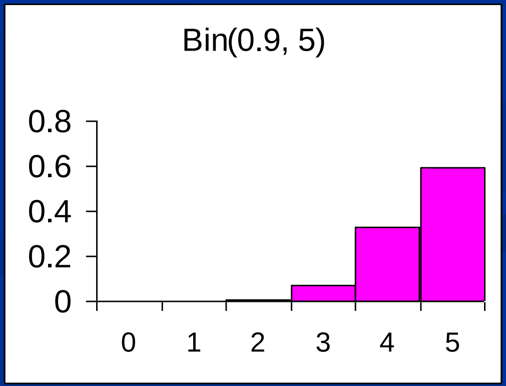
## Overview



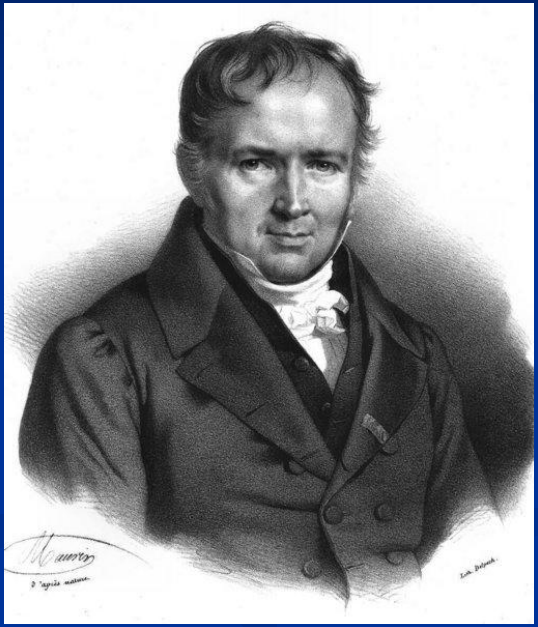



# The Poisson Distribution

## Overview

- When there is a large number of trials, but a small probability of success, binomial calculation becomes impractical
  - Example: Number of deaths from horse kicks in the Army in different years



Simeon D. Poisson (1781-1840)

- The mean number of successes from $n$ trials is $\mu = np$
  - Example: 64 deaths in 20 years from thousands of soldiers

# The Poisson Distribution
## Overview

- If we substitute $\mu/n$ for $p$, and let $n$ tend to infinity, the binomial distribution becomes the Poisson distribution:

$$P(x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

# The Poisson Distribution
## Overview

- Poisson distribution is applied where random events in space or time are expected to occur
- Deviation from Poisson distribution may indicate some degree of non-randomness in the events under study
- Investigation of cause may be of interest

# The Poisson Distribution
## Emission of α-particles

- Rutherford, Geiger, and Bateman (1910) counted the number of α-particles emitted by a film of polonium in 2608 successive intervals of one-eighth of a minute
  - What is $n$?
  - What is $p$?
- Do their data follow a Poisson distribution?

# The Poisson Distribution

## Emission of α-particles

- Calculation of $\mu$:

$$\mu = \text{No. of particles per interval}$$
$$= 10097/2608$$
$$= 3.87$$

- Expected values:

$$2680 \times P(x) = 2608 \times \frac{e^{-3.87}(3.87)^x}{x!}$$

| No. α-particles | Observed |
|---|---|
| 0 | 57 |
| 1 | 203 |
| 2 | 383 |
| 3 | 525 |
| 4 | 532 |
| 5 | 408 |
| 6 | 273 |
| 7 | 139 |
| 8 | 45 |
| 9 | 27 |
| 10 | 10 |
| 11 | 4 |
| 12 | 0 |
| 13 | 1 |
| 14 | 1 |
| Over 14 | 0 |
| Total | 2608 |

# The Poisson Distribution

## Emission of $\alpha$-particles

| No. $\alpha$-particles | Observed | Expected |
|---|---|---|
| 0 | 57 | 54 |
| 1 | 203 | 210 |
| 2 | 383 | 407 |
| 3 | 525 | 525 |
| 4 | 532 | 508 |
| 5 | 408 | 394 |
| 6 | 273 | 254 |
| 7 | 139 | 140 |
| 8 | 45 | 68 |
| 9 | 27 | 29 |
| 10 | 10 | 11 |
| 11 | 4 | 4 |
| 12 | 0 | 1 |
| 13 | 1 | 1 |
| 14 | 1 | 1 |
| Over 14 | 0 | 0 |
| Total | 2608 | 2680 |

# The Poisson Distribution

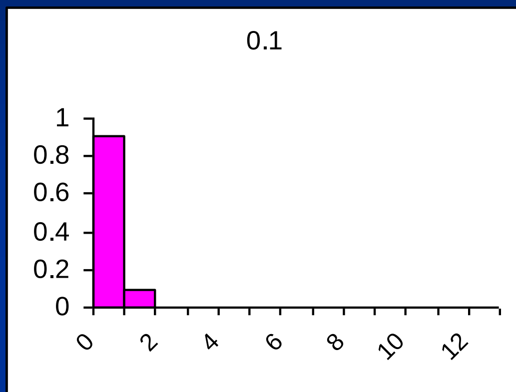
## Emission of α-particles

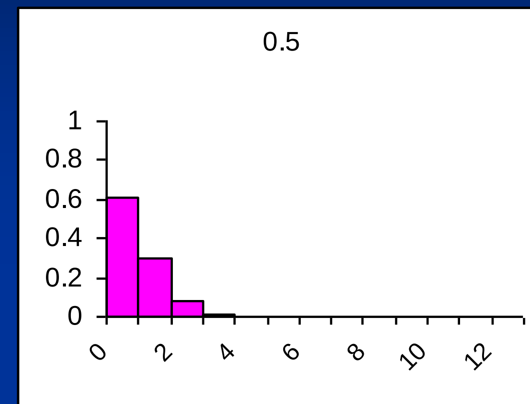
Random events

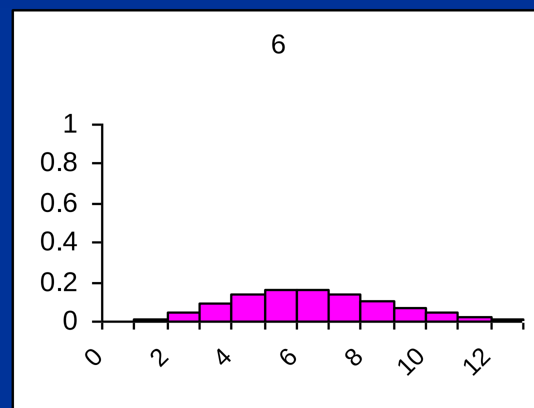
Regular events

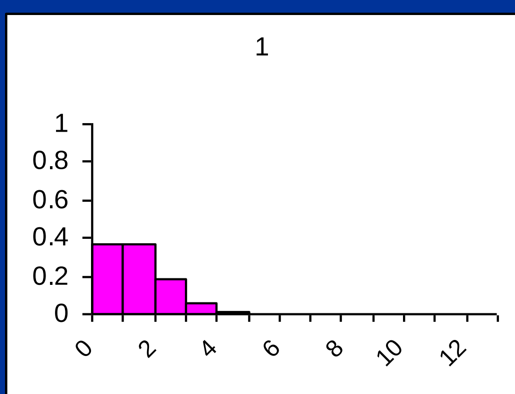
Clumped events

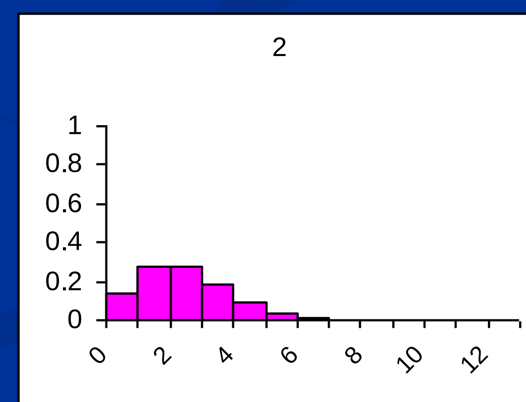
# The Poisson Distribution



# The Expected Value of a Discrete Random Variable

$$E(X) = \sum_{i=1}^{n} a_i p_i = a_1 p_1 + a_2 p_2 + \ldots + a_n p_n$$

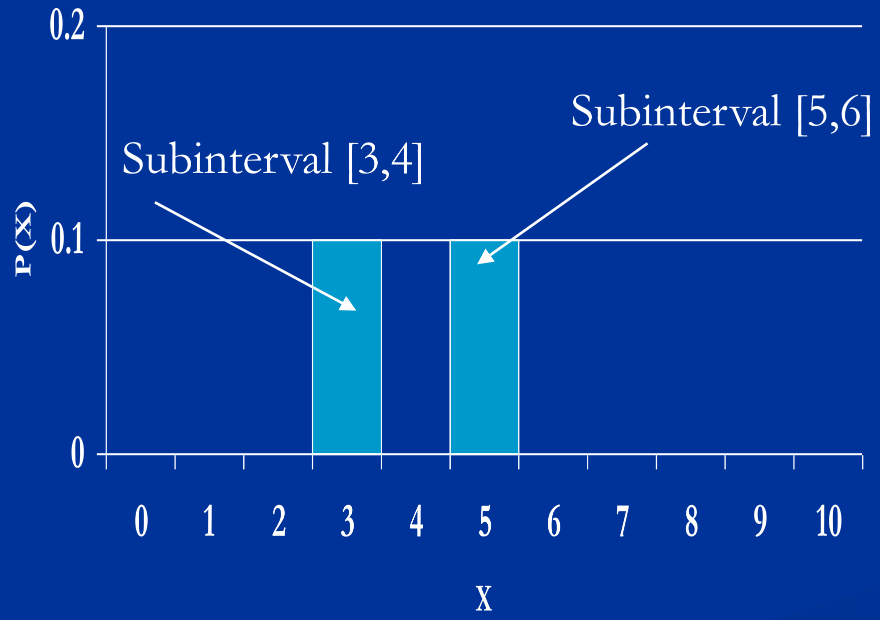
# The Variance of a Discrete Random Variable

$$\sigma^2(X) = E\left[X - E(X)\right]^2$$

$$= \sum_{i=1}^{n} p_i \left( a_i - \sum_{i=1}^{n} a_i p_i \right)^2$$

# Uniform random variables

- The closed unit interval, which contains all numbers between 0 and 1, including the two end points 0 and 1




$$f(x) = \begin{cases} 1/10, 0 \le x \le 10 \\ 0, otherwise \end{cases}$$

The probability density function

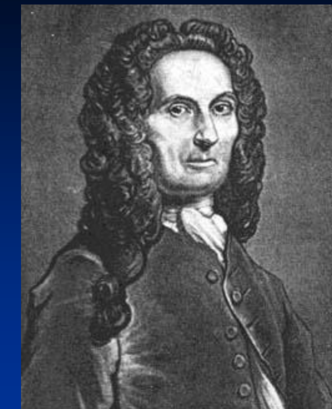
# The Expected Value of a continuous Random Variable

$$E(X) = \int xf(x)dx$$

For an uniform random variable *x*, where *f(x)* is defined on the interval [a,b], and where a<b,

$$E(X) = (b+a)/2 \quad \text{and} \quad \sigma^2(X) = \frac{(b-a)^2}{12}$$

# The Normal Distribution
## Overview



Abraham de Moivre (1667-1754)

- Discovered in 1733 by de Moivre as an approximation to the binomial distribution when the number of trails is large
- Derived in 1809 by Gauss
- Importance lies in the Central Limit Theorem, which states that the sum of a large number of independent random variables (binomial, Poisson, etc.) will approximate a normal distribution
  - Example: Human height is determined by a large number of factors, both genetic and environmental, which are additive in their effects. Thus, it follows a normal distribution.


Karl F. Gauss (1777-1855)

# The Normal Distribution

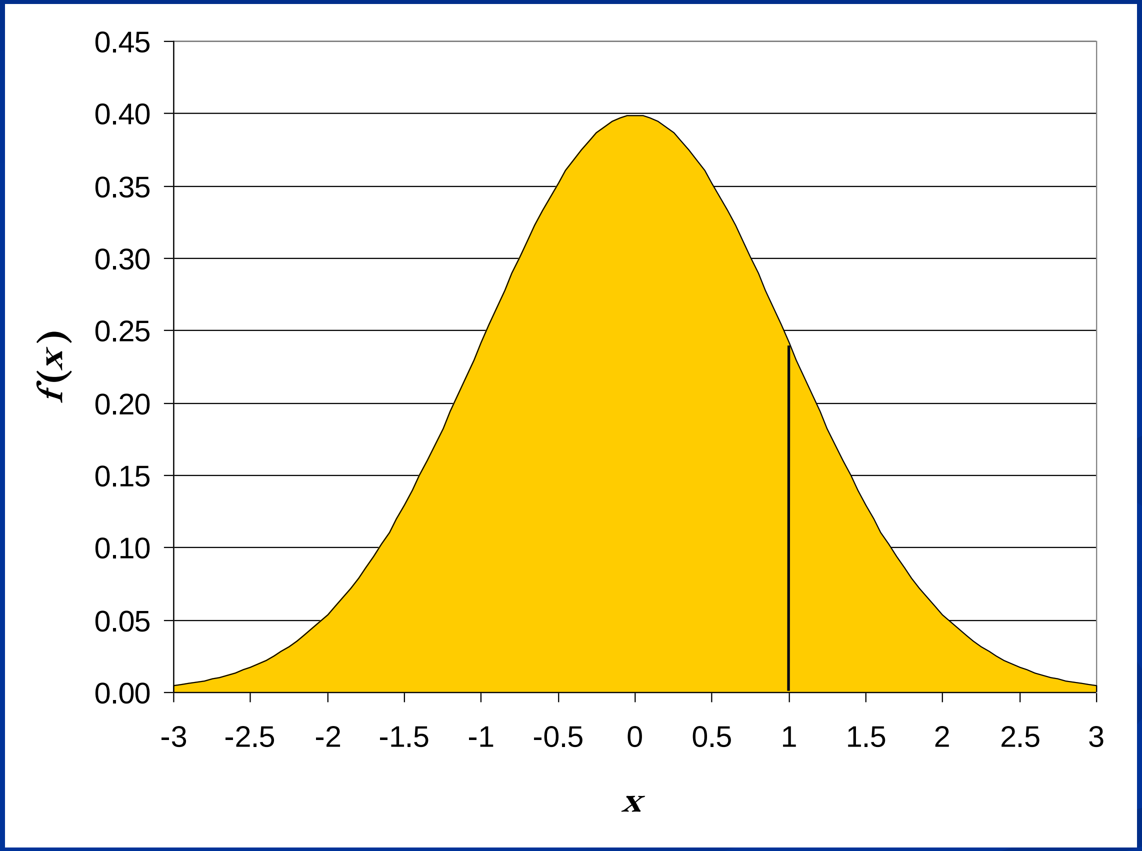
## Overview

- A continuous random variable is said to be normally distributed with mean $\mu$ and variance $\sigma^2$ if its probability density function is

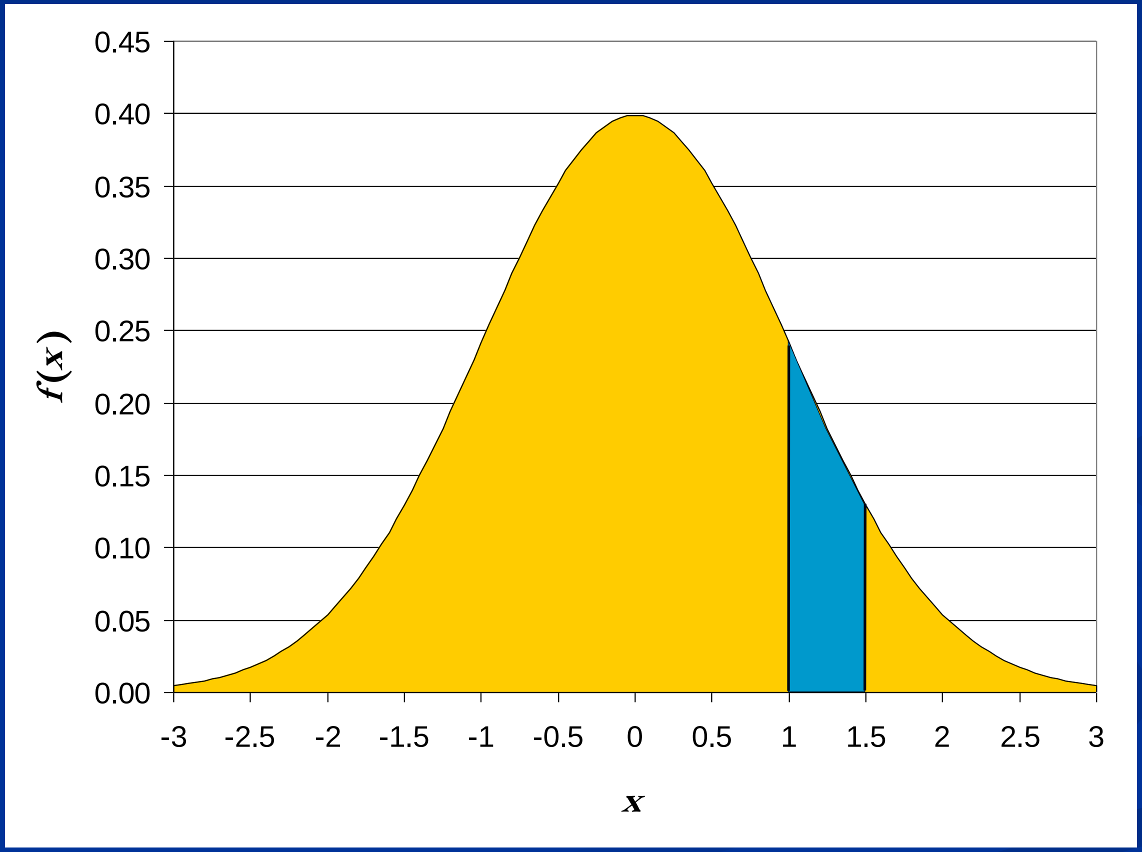
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $f(x)$ is not the same as $P(x)$
  - $P(x)$ would be 0 for every $x$ because the normal distribution is continuous
  - However, $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$

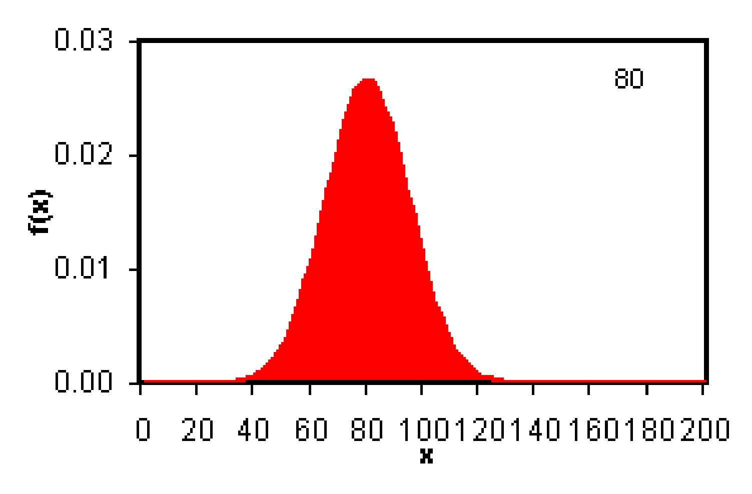
# The Normal Distribution
## Overview



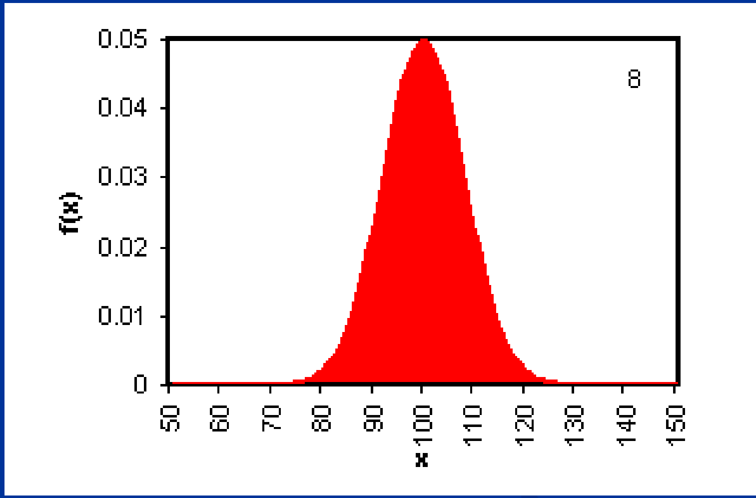
# The Normal Distribution
## Overview



# The Normal Distribution
## Overview




Mean changes




Variance changes

# The Normal Distribution

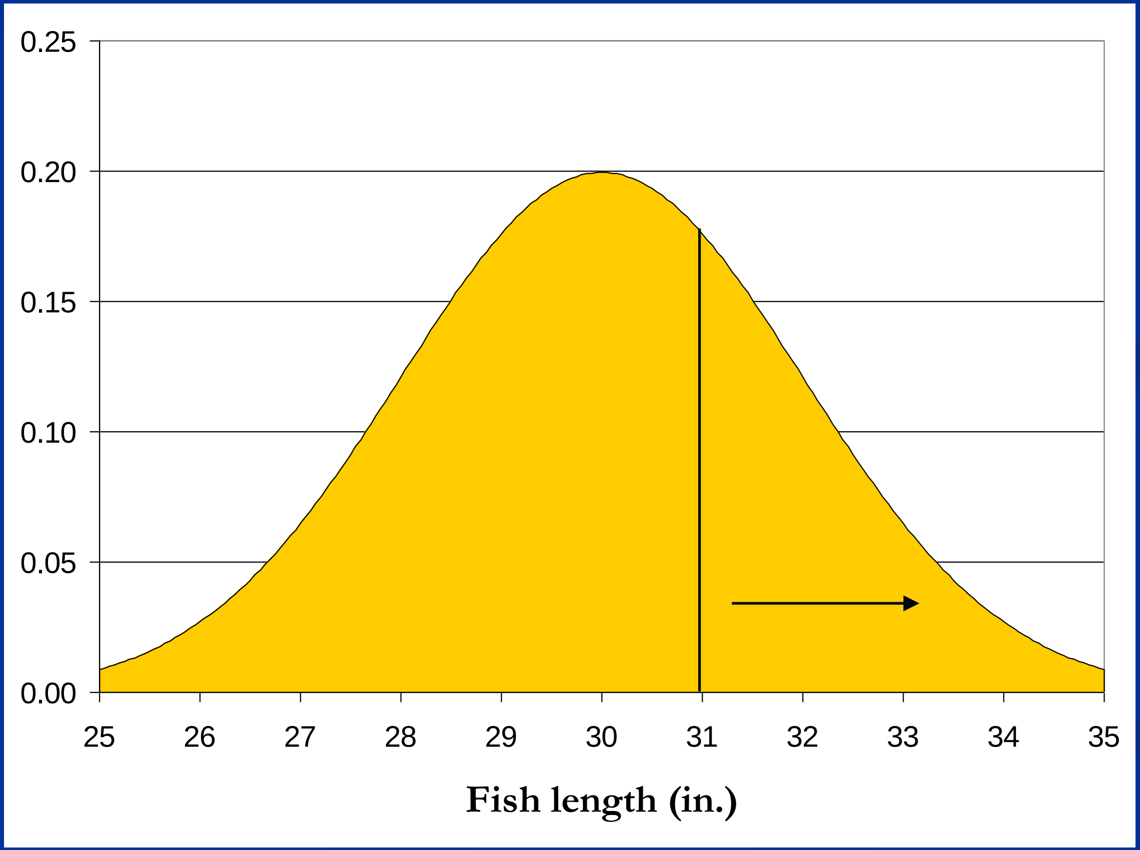
- A sample of rock cod in Monterey Bay suggests that the mean length of these fish is $\mu = 30$ in. and $\sigma^2 = 4$ in.

- Assume that the length of rock cod is a normal random variable

- If we catch one of these fish in Monterey Bay,
  - What is the probability that it will be at least 31 in. long?
  - That it will be no more than 32 in. long?
  - That its length will be between 26 and 29 inches?

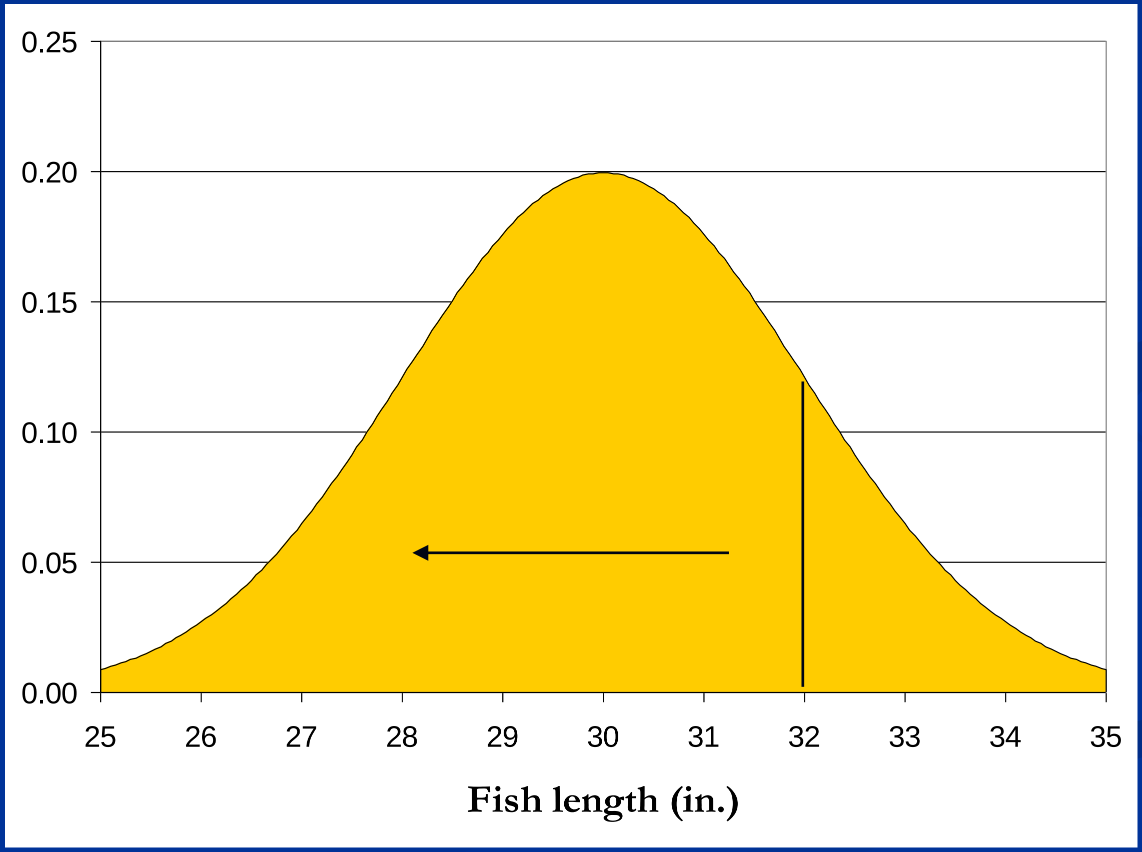
# The Normal Distribution

- What is the probability that it will be at least 31 in. long?



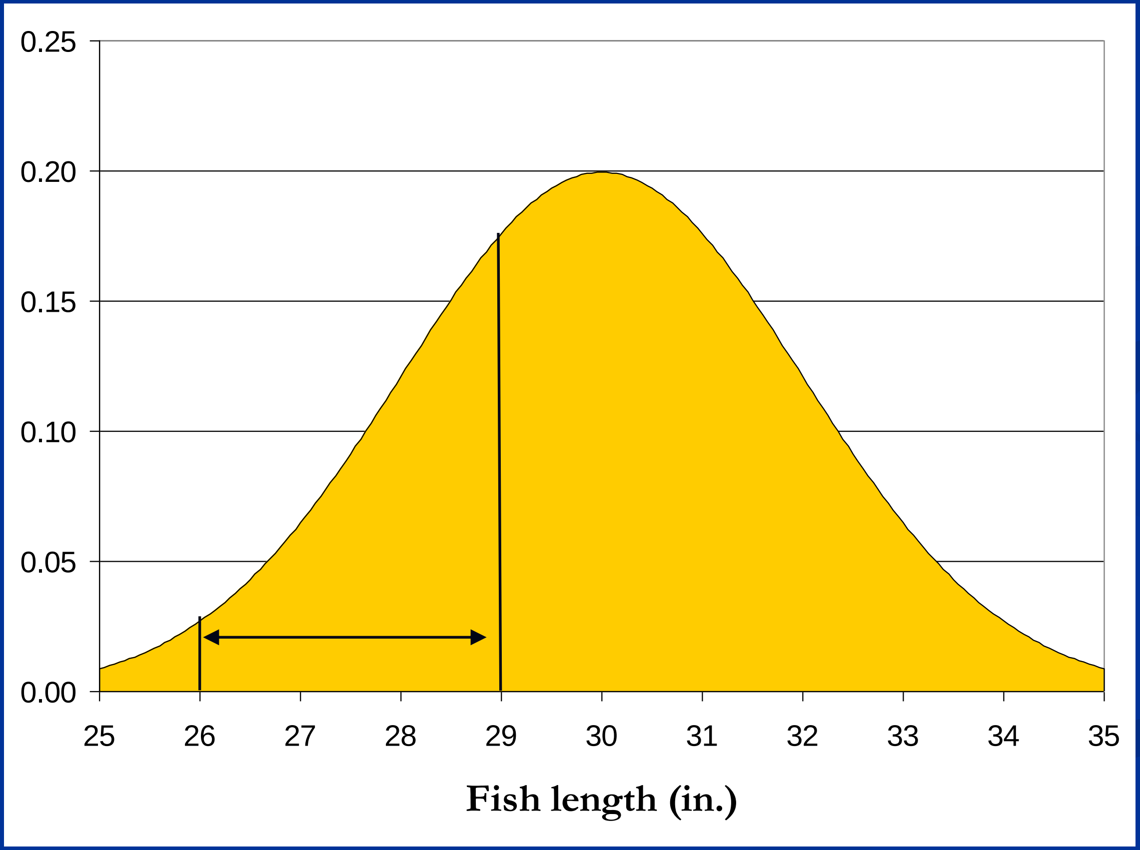
# The Normal Distribution

- That it will be no more than 32 in. long?



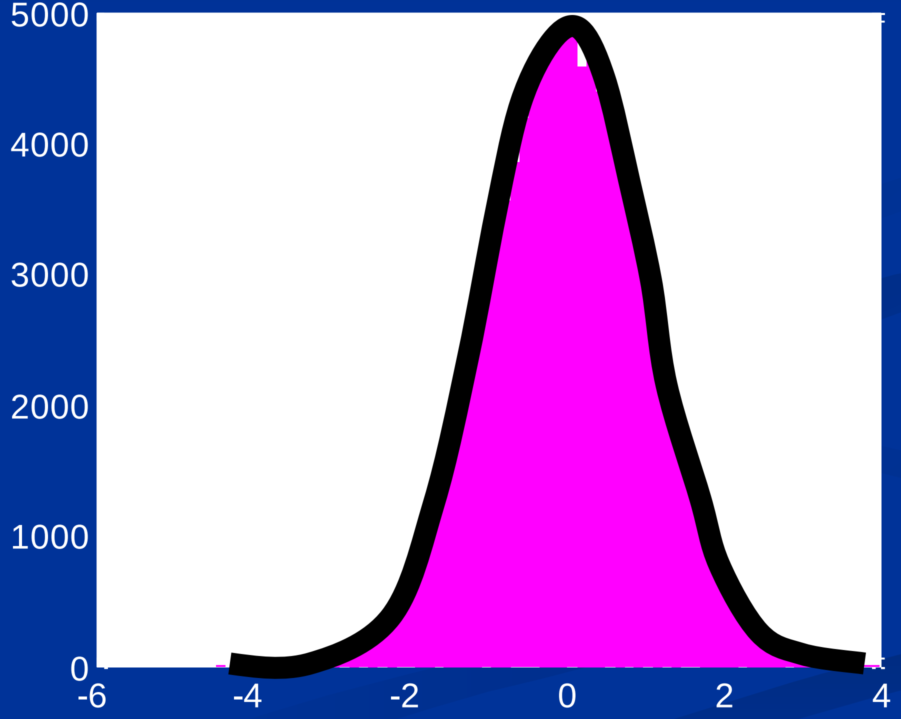
# The Normal Distribution

- That its length will be between 26 and 29 inches?



# Standard Normal Distribution

- $\mu=0$ and $\sigma^2=1$



# Useful properties of the normal distribution

1. The normal distribution has useful properties:

- Can be added $E(X+Y)= E(X)+E(Y)$ and $\sigma 2(X+Y)= \sigma 2(X)+ \sigma 2(Y)$
- Can be transformed with ***shift*** and ***change of scale*** operations

# Consider two random variables X and Y

Let $X \sim N(\mu,\sigma)$ and let $Y=aX+b$ where a and b area constants

*Change of scale* is the operation of multiplying *X* by a constant "***a***" because one unit of X becomes "a" units of Y.

*Shift* is the operation of adding a constant "*b*" to X because we simply move our random variable X "b" units along the x-axis.

If X is a normal random variable, then the new random variable Y created by this operations on X is also a random normal variable

# For $X \sim N(\mu,\sigma)$ and $Y=aX+b$

- $E(Y) = a\mu+b$
- $\sigma^2(Y)=a^2 \sigma^2$
- A special case of a change of scale and shift operation in which $a = 1/\sigma$ and $b = -1(\mu/\sigma)$
- $Y=(1/\sigma)X-\mu/\sigma$
- $Y=(X-\mu)/\sigma$ gives
- $E(Y)=0$ and $\sigma^2(Y) = 1$

# The Central Limit Theorem

- That Standardizing any random variable that itself is a sum or average of a set of independent random variables results in a new random variable that is nearly the same as a standard normal one.
- The only caveats are that the sample size must be large enough and that the observations themselves must be independent and all drawn from a distribution with common expectation and variance.