

Rate of convergence

Two Fundamental Approaches

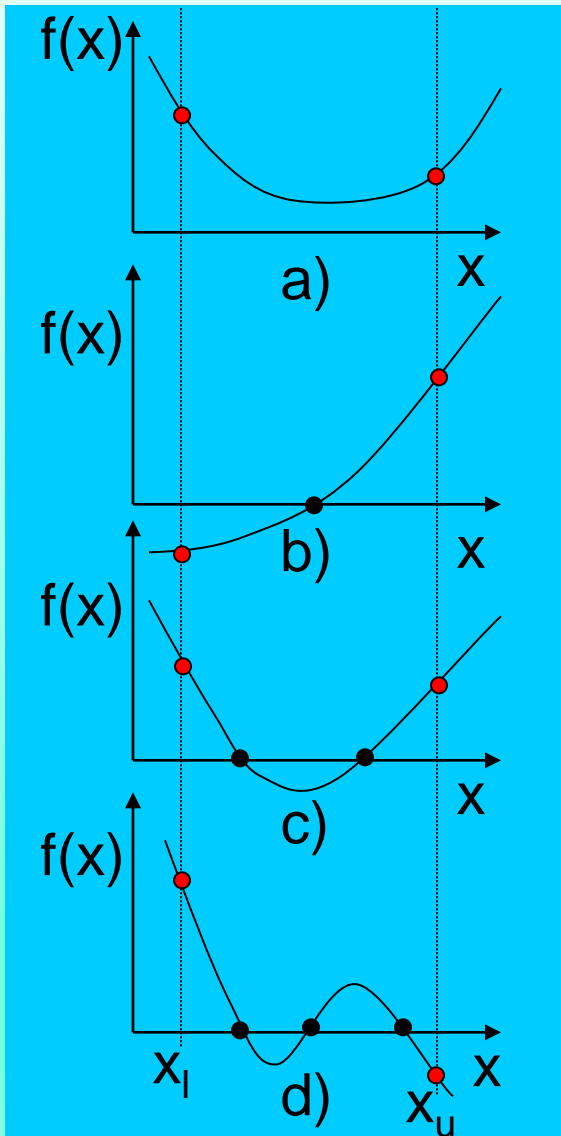
1. *Bracketing or Closed Methods*

- Bisection Method
- False-position Method (*Regula falsi*).

2. *Open Methods*

- Newton-Raphson Method
- Secant Method
- Fixed point Methods

Bracketing Methods



In Figure a) we have the case of $f(x_l)$ and $f(x_u)$ with the same sign, and there is no root in the interval (x_l, x_u) .

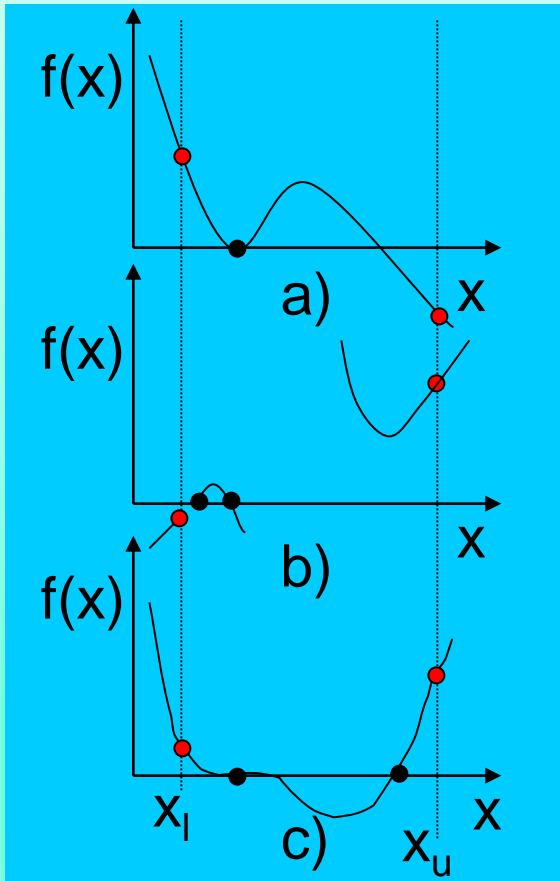
In Figure b) we have the case of $f(x_l)$ and $f(x_u)$ With different sign, and there is a root in the interval (x_l, x_u) .

In Figure c) we have the case of $f(x_l)$ and $f(x_u)$ with the same sign, and there are two roots.

In Figure d) we have the case of $f(x_l)$ and $f(x_u)$ with different sign, and there is an odd number of roots.

Bracketing Methods

- Though the cases above are generally valid, there are cases in which they do not hold.



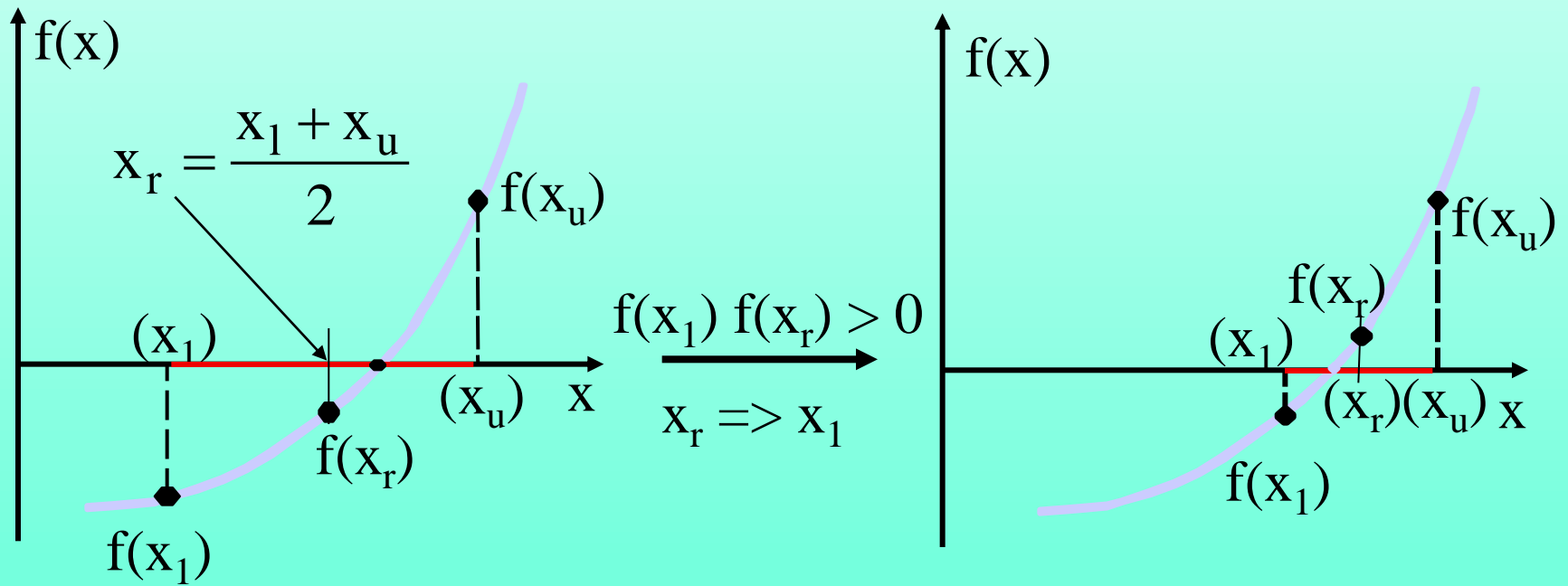
In Figure a) we have the case of $f(x_l)$ and $f(x_u)$ with different sign, but there is a double root.

In Figure b) We have the case of $f(x_l)$ and $f(x_u)$ With different sign, but there are two discontinuities.

In Figure c) we have the case of $f(x_l)$ and $f(x_u)$ with the same sign, but there is a multiple root.

Bracketing Methods (Bisection method)

Bisection Method



Bracketing Methods (Bisection method)

Bisection Method

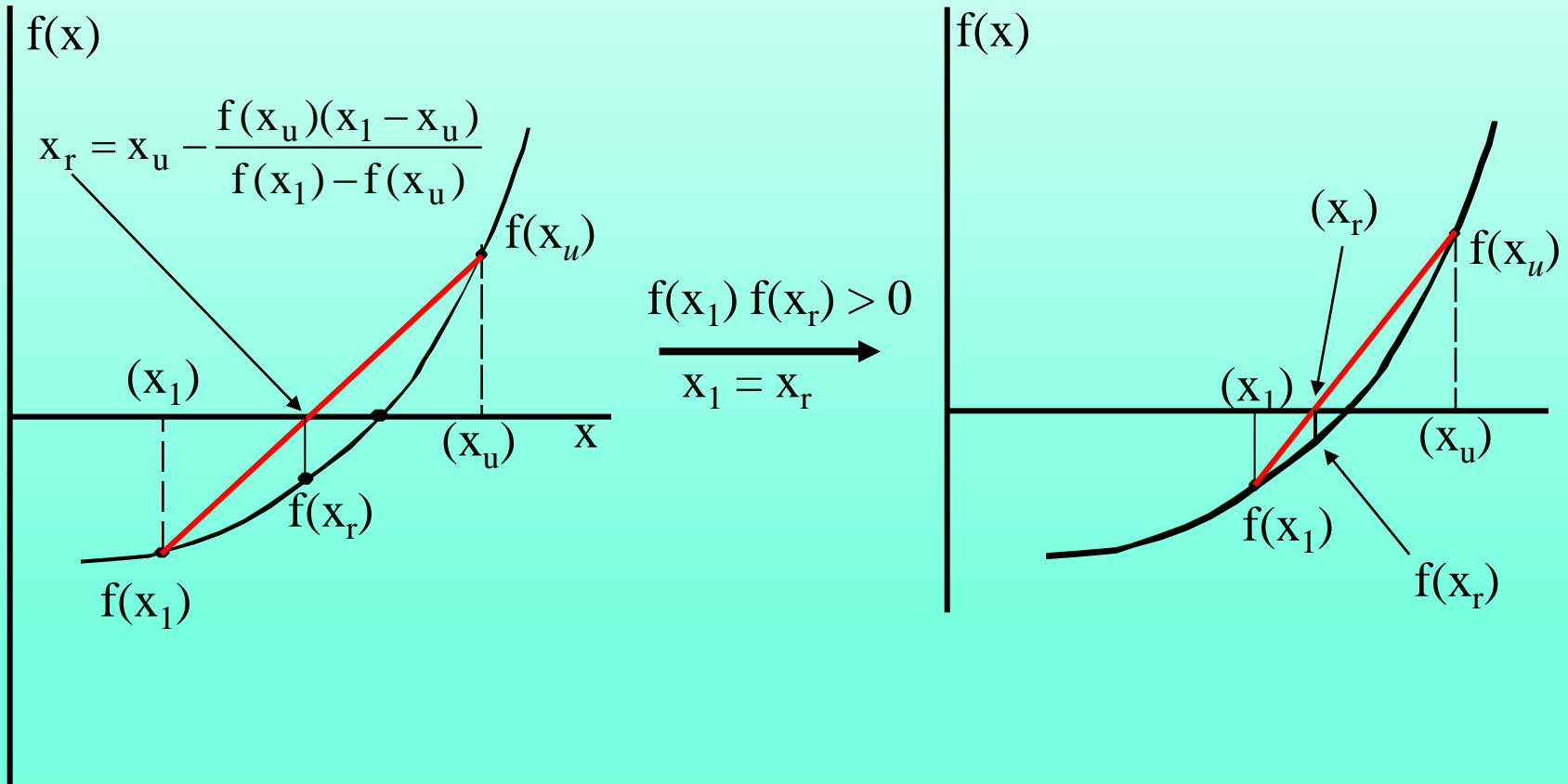
Advantages:

1. Simple
2. Estimate of maximum error: $|E_{\max}| \leq \left| \frac{x_l - x_u}{2} \right|$
3. Convergence guaranteed $|E_{\max}^{i+1}| = 0.5 |E_{\max}^i|$

Disadvantages:

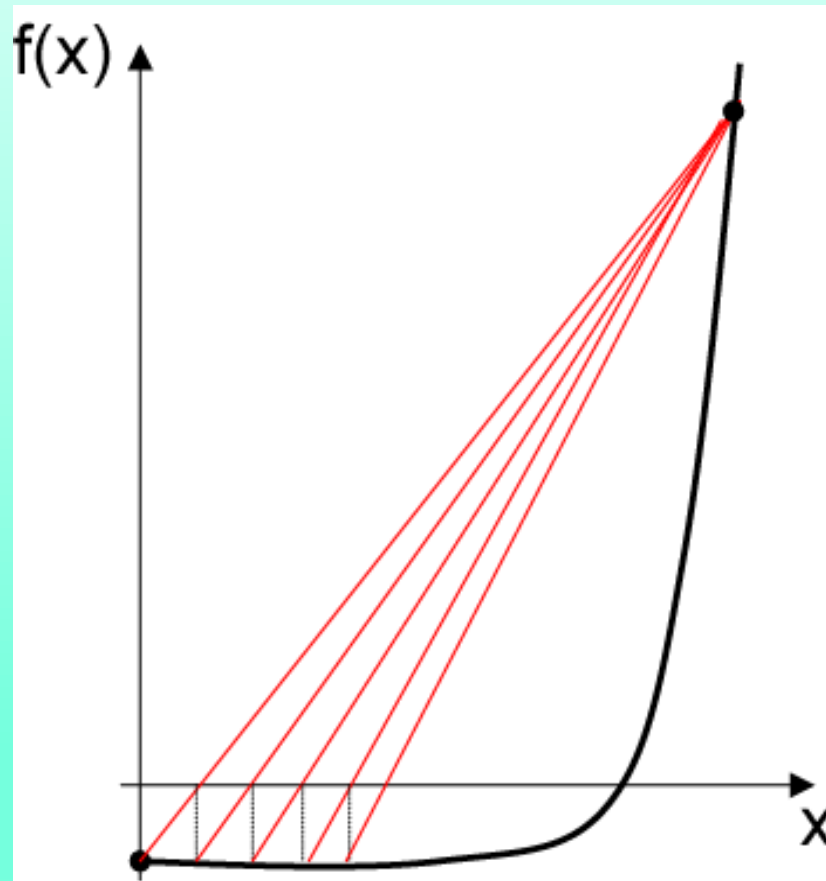
1. Slow
2. Requires two good initial estimates which define an interval around root:
 - use graph of function,
 - incremental search, or
 - trial & error

False-position Method



Bracketing Methods (False-position Method)

There are some cases in which the false position method is very slow, and the bisection method gives a faster solution.



Bracketing Methods (False-position Method)

Summary of False-Position Method:

Advantages:

1. Simple
2. Brackets the Root

Disadvantages:

1. Can be VERY slow
2. Like Bisection, need an initial interval around the root.

Open Methods

Roots of Equations - Open Methods

Characteristics:

1. Initial estimates need not bracket the root
2. Generally converge faster
3. **NOT** guaranteed to converge

Open Methods Considered:

- Fixed-point Methods
- Newton-Raphson Iteration
- Secant Method


Roots of Equations

Two Fundamental Approaches

1. *Bracketing or Closed Methods*

- Bisection Method
- False-position Method

2. *Open Methods*

- One Point Iteration
- **Newton-Raphson Iteration**
-  - **Secant Method**

Open Methods (Newton-Raphson Method)

Newton-Raphson Method:

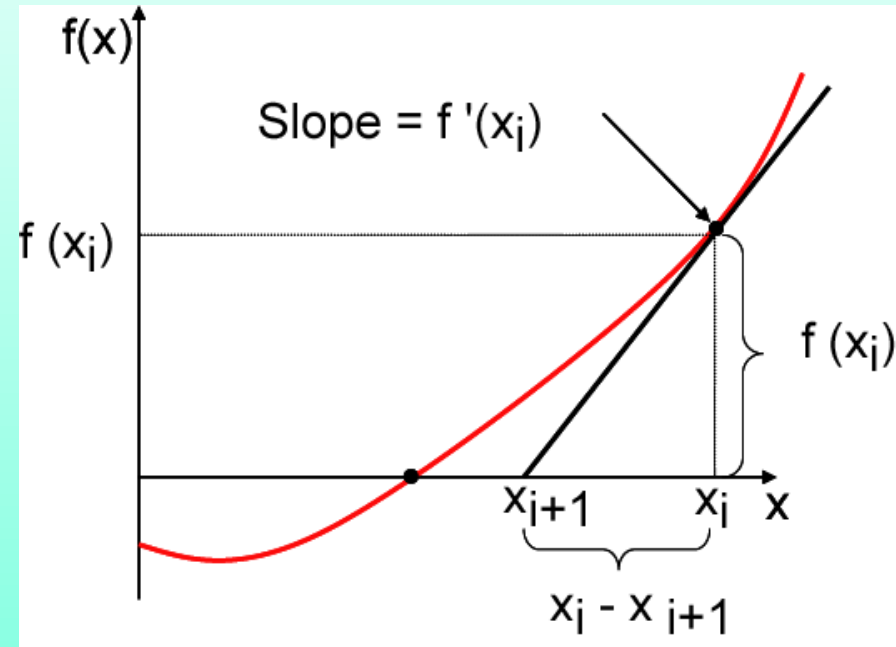
Geometrical Derivation:

Slope of tangent at x_i is

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

Solve for x_{i+1} :

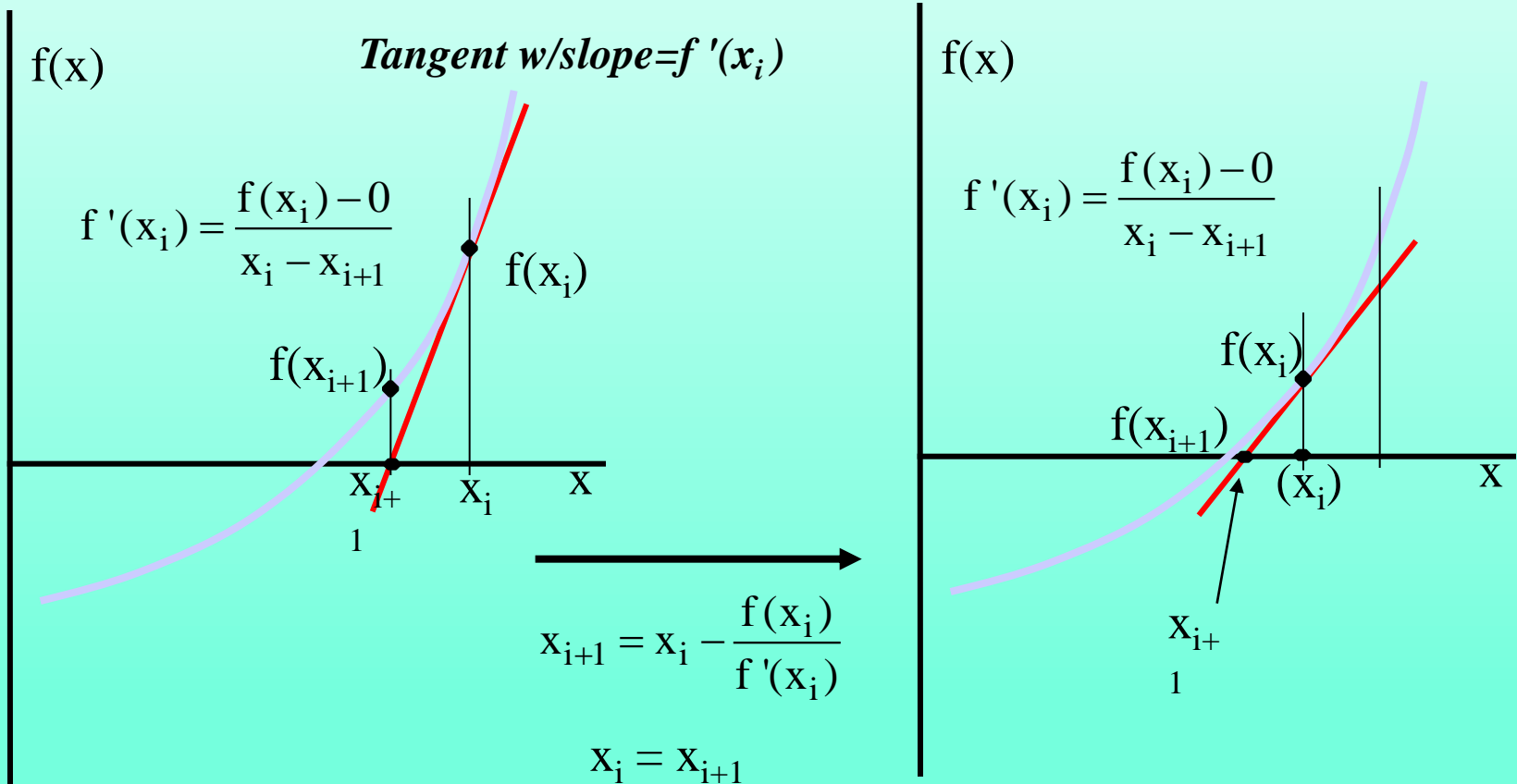
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



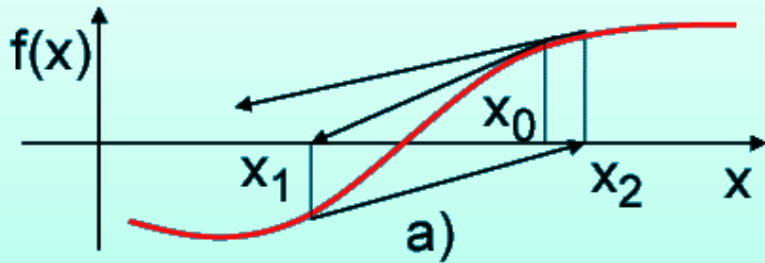
[Note that this is the same form as the generalized one-point iteration, $x_{i+1} = g(x_i)$]

Open Methods (Newton-Raphson Method)

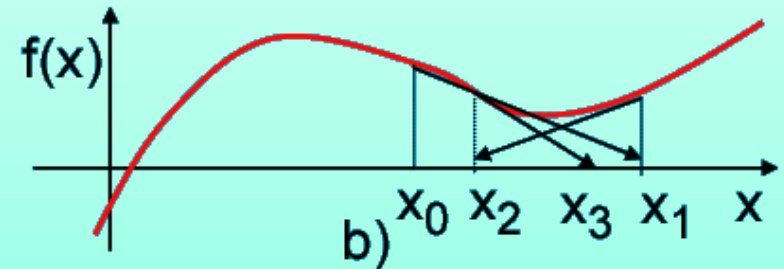
Newton-Raphson Method



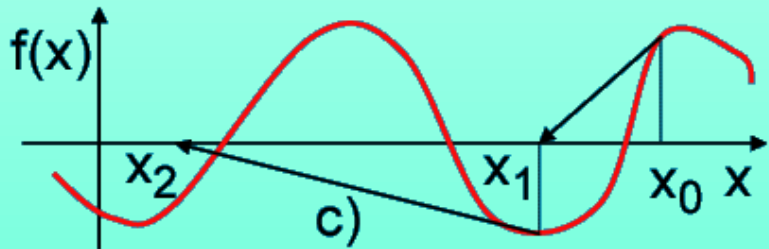
Open Methods



a) Inflection point in the neighborhood of a root.

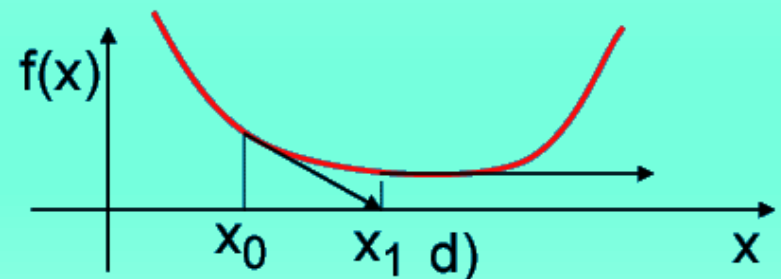


b) Oscillation in the neighborhood of a maximum or minimum.



c) Jumps in functions with several roots.

d) Existence of a null derivative.



Open Methods (Newton-Raphson Method)

Bond Example:

To apply Newton-Raphson method to:

$$f(i) = 7,500 - 1,000 \left[\frac{1 - (1+i)^{-20}}{i} \right] = 0$$

We need the derivative of the function:

$$f'(i) = \frac{1,000}{i} \left\{ \left[\frac{1 - (1+i)^{-20}}{i} \right] - 20(1+i)^{-21} \right\}$$

Rate of convergence

compares the convergence of all the methods.

