The Finite Element Method Defined

The Finite Element Method (FEM) is a weighted residual method that uses compactly-supported basis functions.

Introduction to FEM

Unit 5

Brief Comparison with Other Methods

Finite Difference (FD) Method:

FD approximates an *operator* (e.g., the derivative) and solves a problem on a set of *points* (the grid)

Finite Element (FE) Method:

FE uses exact operators but approximates the *solution basis functions*. Also, FE solves a problem on the *interiors* of grid cells (and optionally on the gridpoints as well).

Brief Comparison with Other Methods

Spectral Methods:

Spectral methods use global basis functions to approximate a solution across the entire domain.

Finite Element (FE) Method:

FE methods use compact basis functions to approximate a solution on individual elements.

Overview of the Finite Element Method

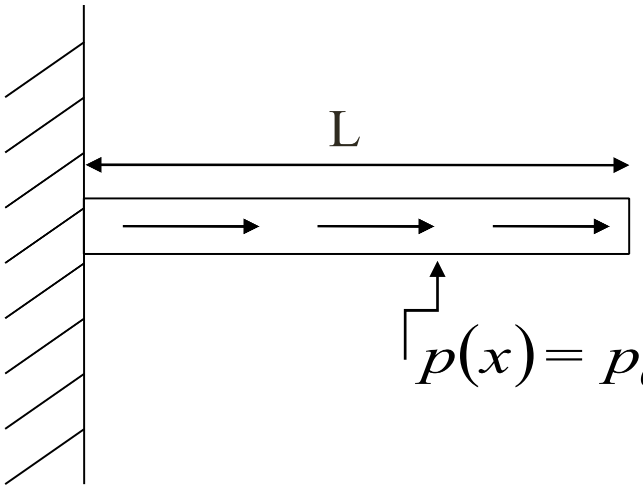
$$(S) \Leftrightarrow (W) \approx (G) \Leftrightarrow (M)$$

Strong form	Weak form	Galerkin approx.	Matrix form

Sample Problem

Axial deformation of a bar subjected to a uniform load

(1-D Poisson equation)



$$p(x) = p_0$$

$$EA\frac{d^2u}{dx^2} = p_0$$

$$u(0) = 0$$

$$EA\frac{du}{dx}\bigg|_{x=L} = 0$$

$$x = [0, L]$$

u = axial displacement

E=Young's modulus = 1

A=Cross-sectional area = 1

Strong Form

The set of governing PDE’s, with boundary conditions, is called the “strong form” of the problem.

Hence, our strong form is (Poisson equation in 1-D):

$$\frac{d^2u}{dx^2} = p_0$$

$$u(0) = 0$$

$$\left.\frac{du}{dx}\right|_{x=L} = 0$$

Weak Form

We now reformulate the problem into the weak form.

The weak form is a *variational statement* of the problem in which we integrate against a *test function*. The choice of test function is up to us.

This has the effect of relaxing the problem; instead of finding an exact solution everywhere, we are finding a solution that satisfies the strong form on average over the domain.

Weak Form

$$\frac{d^2u}{dx^2} = p_0$$

Strong Form

$$\frac{d^2u}{dx^2} - p_0 = 0$$

Residual ***R****=0*

$$\int_0^L \left(\frac{d^2u}{dx^2} - p_0 \right) v\,dx = 0$$

Weak Form

v is our test function

We will choose the test function later.

Weak Form

Why is it "weak"?

It is a weaker statement of the problem.

A solution of the strong form will also satisfy the weak form, but not vice versa.

Analogous to "weak" and "strong" convergence:

$$\text{strong}: \lim_{n \to \infty} x_n = x$$

$$\text{weak}: \lim_{n \to \infty} \langle f | x_n \rangle = \langle f | x \rangle \ \forall f$$

Weak Form

Choosing the test function:

We can choose any *v* we want, so let's choose *v* such that it satisfies *homogeneous* boundary conditions wherever the actual solution satisfies *Dirichlet* boundary conditions. We'll see why this helps us, and later will do it with more mathematical rigor.

So in our example, $u(0)=0$ so let $v(0)=0$.

Weak Form

Returning to the weak form:

$$\int_0^L \left(\frac{d^2u}{dx^2} - p_0 \right) v\,dx = 0$$

$$\int_0^L \frac{d^2u}{dx^2} v\,dx = \int_0^L p_0 v\,dx$$

Integrate LHS by parts:

$$= -\int_0^L \frac{du}{dx}\frac{dv}{dx}dx + \left[v(x)\frac{du}{dx} \right]_{x=0}^{x=L}$$

$$= -\int_0^L \frac{du}{dx}\frac{dv}{dx}dx + v(L)\frac{du}{dx}\bigg|_{x=L} - v(0)\frac{du}{dx}\bigg|_{x=0}$$

Weak Form

Recall the boundary conditions on u and v:

$$u(0) = 0$$

$$\left.\frac{du}{dx}\right|_{x=L} = 0$$

$$v(0) = 0$$

Hence,

$$-\int_0^L \frac{du}{dx}\frac{dv}{dx}dx + v(L)\left.\frac{du}{dx}\right|_{x=L} - v(0)\left.\frac{du}{dx}\right|_{x=0}$$

$$\int_0^L \frac{du}{dx}\frac{dv}{dx}dx = \int_0^L p_0 v\,dx$$

The weak form satisfies Neumann conditions automatically!

Weak Form

Why is it "variational"?

$$\int_0^L \frac{du}{dx}\frac{dv}{dx}dx = \int_0^L p_0 v dx$$

Variational statement:

Find $u \in H^1$ such that $B(u,v) = F(v) \quad \forall v \in H_0^1$

B a bilinear functional, F a linear functional

u and v are functions from an infinite-dimensional function space H

Galerkin's Method

We still haven't done the "finite element method" yet, we have just restated the problem in the weak formulation.

So what makes it "finite elements"?

Solving the problem locally on elements

Finite-dimensional approximation to an infinite- dimensional space → **Galerkin's Method**

Galerkin's Method

Choose finite basis $\{\varphi_i\}_{i=i}^{N}$

Then,

$$u(x) = \sum_{j=1}^{N} c_j \varphi_j(x), \qquad c_j \text{ unkowns to solve for}$$

$$v(x) = \sum_{j=1}^{N} b_j \varphi_j(x), \qquad b_j \text{ arbitrarily chosen}$$

Insert these into our weak form:

$$\int_0^L \frac{du}{dx}\frac{dv}{dx}dx = \int_0^L p_0 v\, dx$$

$$\int_0^L \sum_{j=1}^{N} c_j \frac{d\varphi_j}{dx}(x) \sum_{i=1}^{N} b_i \frac{d\varphi_i}{dx}(x) dx = \int_0^L p_0 \sum_{j=1}^{N} b_j \varphi_j(x) dx$$

Galerkin's Method

$$\int_0^L \sum_{j=1}^{N} c_j \frac{d\varphi_j}{dx}(x) \sum_{i=1}^{N} b_i \frac{d\varphi_i}{dx}(x) dx = \int_0^L p_0 \sum_{j=1}^{N} b_j \varphi_j(x) dx$$

Rearrangin g :

$$\sum_{i=1}^{N} b_i \sum_{j=1}^{N} c_j \int_0^L \frac{d\varphi_j}{dx} \frac{d\varphi_i}{dx} dx = \sum_{i=1}^{N} b_i \int_0^L p_0 \varphi_i dx$$

Cancelling :

$$\sum_{j=1}^{N} c_j \int_0^L \frac{d\varphi_j}{dx} \frac{d\varphi_i}{dx} dx = \int_0^L p_0 \varphi_i dx$$

Galerkin's Method

$$\sum_{j=1}^{N} c_j \int_0^L \frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx}\,dx = \int_0^L p_0 \varphi_i \,dx$$

We now have a matrix problem $\mathbf{Kc} = \mathbf{F}$, where c_j

is a vector of unknowns,

$$K_{ij} = \int_0^L \frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx}\,dx,$$

and $F_i = \int_0^L p_0 \varphi_i \,dx$

We can already see K_{ij} will be symmetric since we can

interchange i, j without effect.

Galerkin's Method

So what have we done so far?

1) Reformulated the problem in the weak form.

2) Chosen a finite-dimensional approximation to the solution.

Recall weak form written in terms of residual:

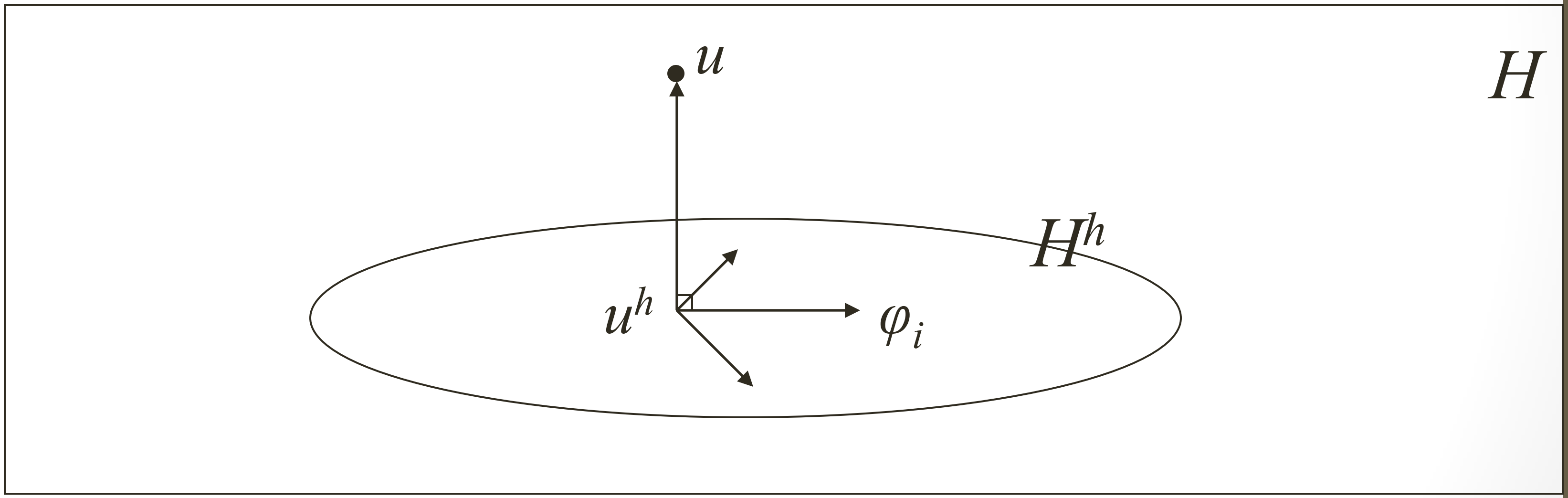
$$\int_0^L \left(\frac{d^2u}{dx^2} - p_0 \right) v\,dx = \int_0^L \mathbf{R} v\,dx = \sum_i b_i \int_0^L \mathbf{R} \varphi_i\,dx = 0$$

This is an L_2 inner-product. Therefore, the residual is orthogonal to our space of basis functions. "Orthogonality Condition"

Orthogonality Condition

$$\int_0^L \left(\frac{d^2 u}{dx^2} - p_0 \right) v\,dx = \int_0^L \mathbf{R} v\,dx = \sum_i b_i \int_0^L \mathbf{R} \varphi_i\,dx = 0$$

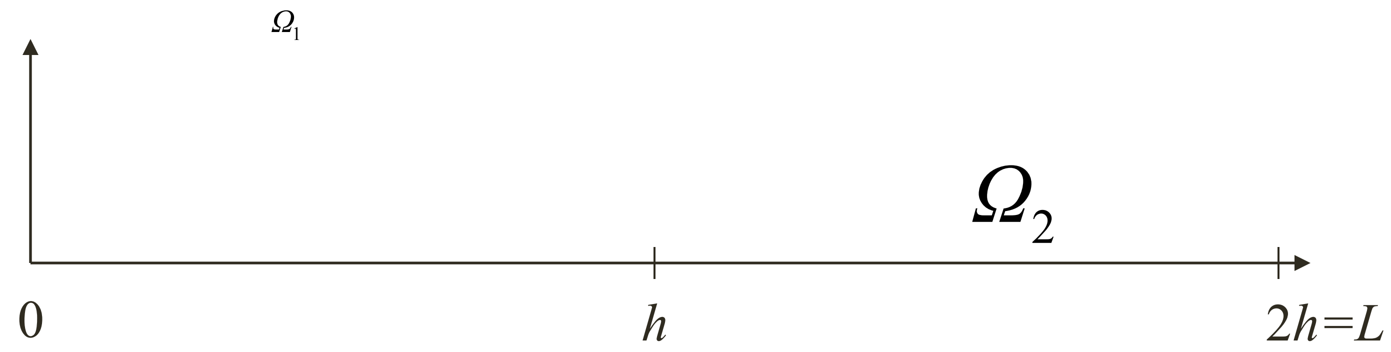
The residual is orthogonal to our space of basis functions:



Therefore, given some space of approximate functions H^h, we are finding u^h that is closest (as measured by the L_2 inner product) to the actual solution u.

Discretization and Basis Functions

Let's continue with our sample problem. Now we discretize our domain. For this example, we will discretize x=[0, L] into 2 "elements".

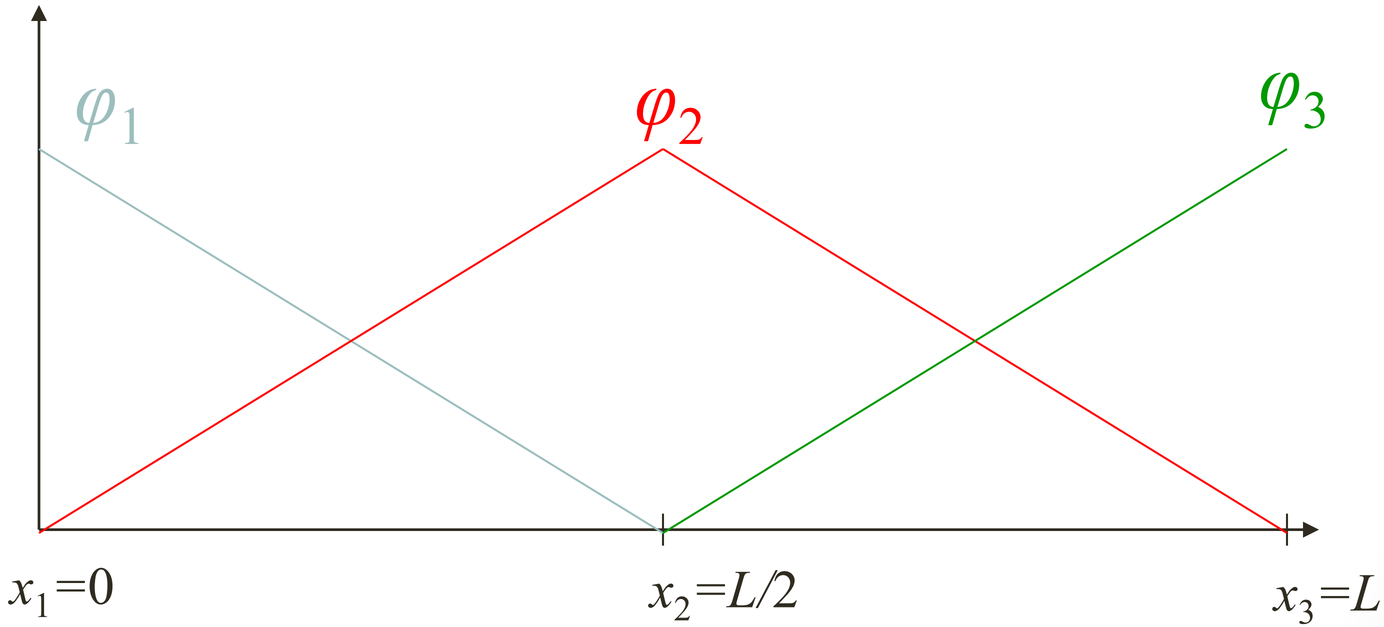


In 1-D, elements are segments. In 2-D, they are triangles, tetrads, etc. In 3-D, they are solids, such as tetrahedra. We will solve the Galerkin problem on each element.

Discretization and Basis Functions

For a set of basis functions, we can choose anything. For simplicity here, we choose piecewise linear "hat functions".

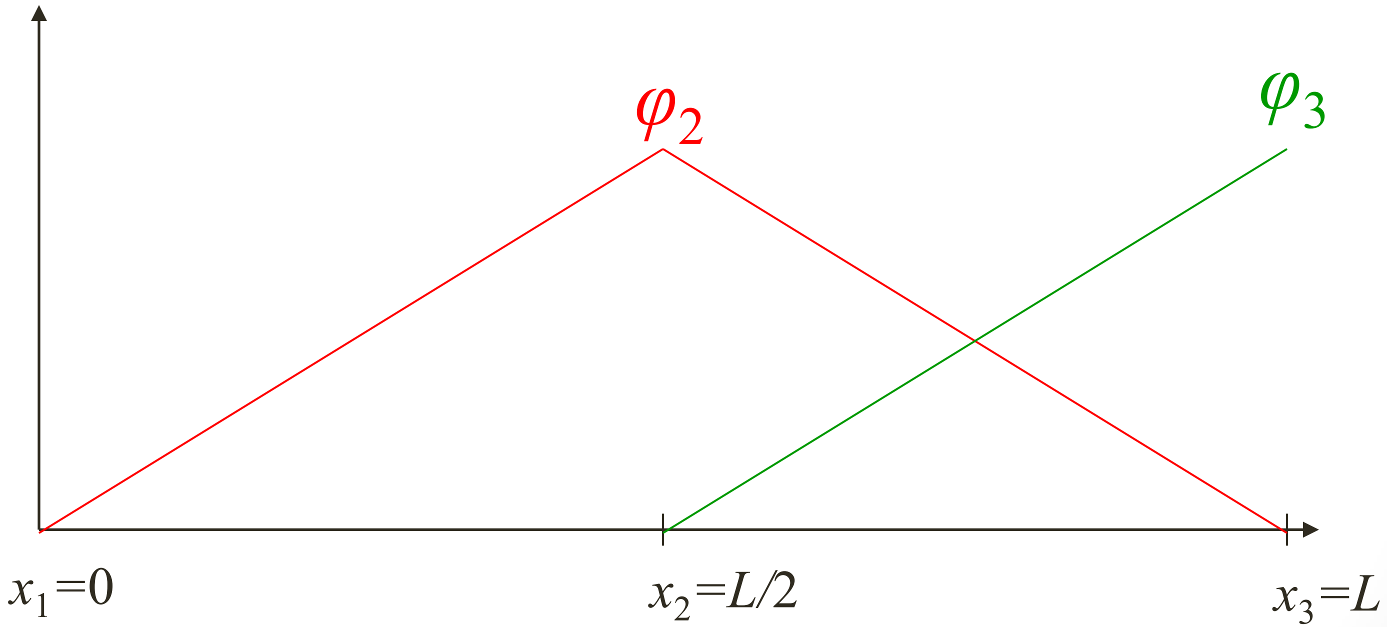
Our solution will be a linear combination of these functions.



Basis functions satisfy: $\varphi_i(x_j) = \delta_j^i \Rightarrow$ Our solution will be interpolatory. Also, they satisfy the partition of unity.

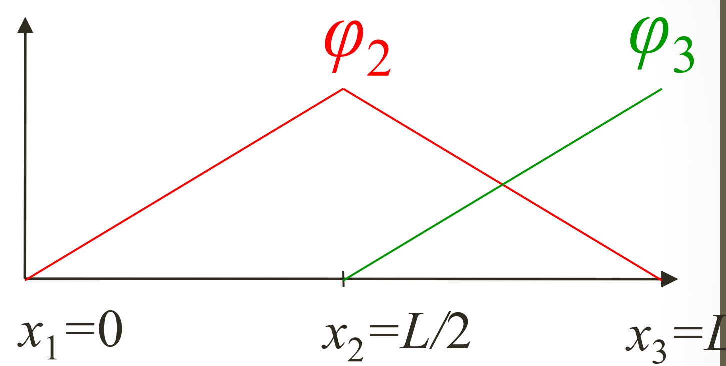
Discretization and Basis Functions

To save time, we can throw out φ_1 a priori because, since in this example $u(0)=0$, we know that the coefficent c_1 must be 0.



Basis Functions

$$\varphi_2 = \begin{cases} \dfrac{2x}{L} & \text{if } x \in \left[0, \frac{L}{2}\right] \\ 2 - \dfrac{2x}{L} & \text{if } x \in \left[\frac{L}{2}, L\right] \\ 0 & \text{otherwise} \end{cases}$$



$$\varphi_3 = \begin{cases} \dfrac{2x}{L} - 1 & \text{if } x \in \left[\frac{L}{2}, L\right] \\ 0 & \text{otherwise} \end{cases}$$

Matrix Formulation

Given our matrix problem $\mathbf{Kc} = \mathbf{F}$:

$$\sum_{j=1}^{N} \underbrace{c_j}_{\mathbf{c}} \underbrace{\int_0^L \frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx}\,dx}_{\mathbf{K}} = \underbrace{\int_0^L p_0\varphi_i\,dx}_{\mathbf{F}} \Rightarrow \mathbf{Kc} = \mathbf{F}$$

We can insert the φ_i chosen on the previous slide and arrive at a linear algebra problem. Differentiating the basis functions, then evaluating the integrals, we have:

$$\mathbf{K} = \frac{1}{L}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}, \quad \mathbf{F} = \frac{p_0}{L}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

In a computer code, differentiating the basis functions can be done in advance, since the basis functions are known, and integration would be performed numerically by quadrature.

It is standard in FEM to use Gaussian quadrature, since it is exact for polynomials.

Notice $\mathbf{K}$ is symmetric as expected.

Solution

Solving the Gaussian elimination problem on the previous slide, we obtain our coefficients c_i :

$\mathbf{c} = \begin{bmatrix} \frac{3p_0L^2}{8} \\ \frac{p_0L^2}{2} \end{bmatrix}$, which when multiplied by basis functions φ_i gives

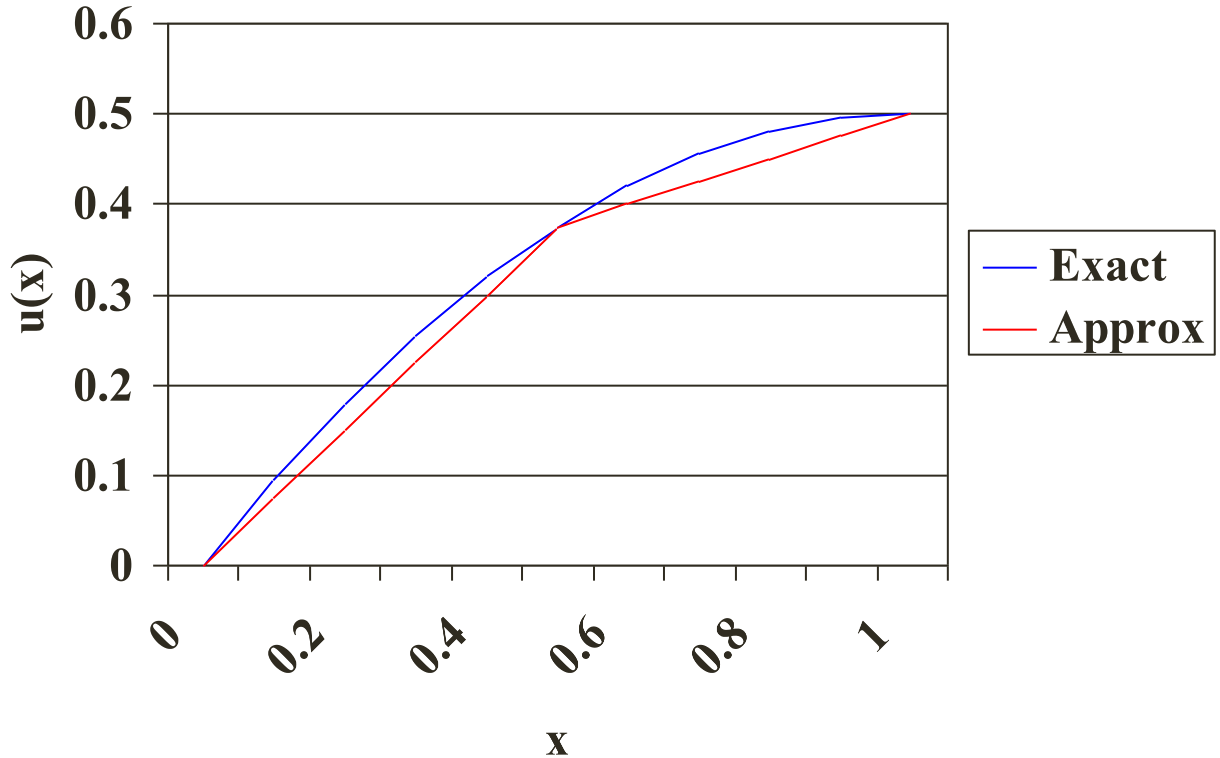
our final numerical solution :

$$\varphi(x) = \begin{cases} \frac{3}{4} p_0 L x & \text{when } x \in \left[0, \frac{L}{2}\right] \\ \frac{1}{4} p_0 (L^2 + Lx) & \text{when } x \in \left[\frac{L}{2}, L\right] \end{cases}$$

The exact analytical solution for this problem is :

$$u(x) = p_0 L x - \frac{p_0 x^2}{2}$$

Solution



Notice the numerical solution is “interpolatory”, or nodally exact.

Concluding Remarks

- Because basis functions are compact, matrix **K** is typically tridiagonal or otherwise sparse, which allows for fast solvers that take advantage of the structure (regular Gaussian elimination is O(N^3), where N is number of elements!). Memory requirements are also reduced.

- Continuity between elements not required. "Discontinuous Galerkin" Method