# *Linear Programming*

## *Operations Management - 5th Edition*

# Lecture Outline

- Model Formulation
- Graphical Solution Method
- Linear Programming Model
- Solution
- Solving Linear Programming Problems with Excel
- Sensitivity Analysis

# Linear Programming (LP)

A model consisting of linear relationships representing a firm's objective and resource constraints

LP is a mathematical modeling technique used to determine a level of operational activity in order to achieve an objective, subject to restrictions called constraints

# Types of LP

| Linear Programming Model Type | OM Application |
|---|---|
| Aggregate Production Planning | Determines the resource capacity needed to meet demand over an immediate time horizon, including units produced, workers hired and fired and inventory. (See Chapter 13.) |
| Product Mix | Mix of different products to produce that will maximize profit or minimize cost given resource constraints such as material, labor, budget, etc. |
| Transportation | Logistical flow of items (goods or services) from sources to destinations, for example, truckloads of goods from plants to warehouses. (See Supplement 10.) |
| Transshipment | Flow of items from sources to destinations with intermediate points, for example shipping from plant to distribution center and then to stores. (See Supplement 10.) |

# Types of LP (cont.)

| Linear Programming Model Type | OM Application |
|---|---|
| Assignment | Assigns work to limited resources, called "Loading," for example, assigning jobs or workers to different machines. (See Chapter 16.) |
| Multiperiod Scheduling | Schedules regular and overtime production, plus inventory to carry over, to meet demand in future periods. |
| Blend | Determines "recipe" requirements, for example, how to blend different petroleum components to produce different grades of gasoline and other petroleum products. |
| Diet | Menu of food items that meets nutritional or other requirements, for example, hospital or school cafeteria menus. |
| Investment/Capital Budgeting | Financial model that determines amount to invest in different alternatives given return objectives and constraints for risk, diversity, etc., for example, how much to invest in new plant, facilities or equipment. |

# Types of LP (cont.)

| Linear Programming Model Type | OM Application |
|---|---|
| Data Envelopment Analysis (DEA) | Compares service units of the same type—banks, hospitals, schools—based on their resources and outputs to see which units are less productive or inefficient. |
| Shortest Route | Shortest routes from sources to destinations, for example, the shortest highway truck route from coast to coast. |
| Maximal Flow | Maximizes the amount of flow from sources to destinations, for example, the flow of work-in process through an assembly operation. |
| Trim-Loss | Determines patterns to cut sheet items to minimize waste, for example, cutting lumber, film, cloth, glass, etc. |
| Facility Location | Selects facility locations based on constraints such as fixed, operating, and shipping costs, production capacity, etc. |
| Set Covering | Selection of facilities that can service a set of other facilities, for example, the selection of distribution hubs that will be able to deliver packages to a set of cities. |

# LP Model Formulation

- Decision variables
  - mathematical symbols representing levels of activity of an operation
- Objective function
  - a linear relationship reflecting the objective of an operation
  - most frequent objective of business firms is to *maximize profit*
  - most frequent objective of individual operational units (such as a production or packaging department) is to *minimize cost*
- Constraint
  - a linear relationship representing a restriction on decision making

# LP Model Formulation (cont.)

Max/min $z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\ \vdots \\ a_{m1}x1 + a_{m2}x_2 + ... + a_{mn}x_n \ (\leq, =, \geq) \ b_m \end{cases}$$

$x_j$ = decision variables

$b_i$ = constraint levels

$c_j$ = objective function coefficients

$a_{ij}$ = constraint coefficients

# LP Model: Example

| | RESOURCE REQUIREMENTS | | |
|---|---|---|---|
| PRODUCT | *Labor (hr/unit)* | *Clay (lb/unit)* | *Revenue ($/unit)* |
| Bowl | 1 | 4 | 40 |
| Mug | 2 | 3 | 50 |

There are 40 hours of labor and 120 pounds of clay available each day

Decision variables

$x_1$ = number of bowls to produce

$x_2$ = number of mugs to produce

# LP Formulation: Example

Maximize $Z = \$40\ x_1 + 50\ x_2$

Subject to

$x_1 + 2x_2 \leq 40$ hr (labor constraint)

$4x_1 + 3x_2 \leq 120$ lb (clay constraint)

$x_1, x_2 \geq 0$

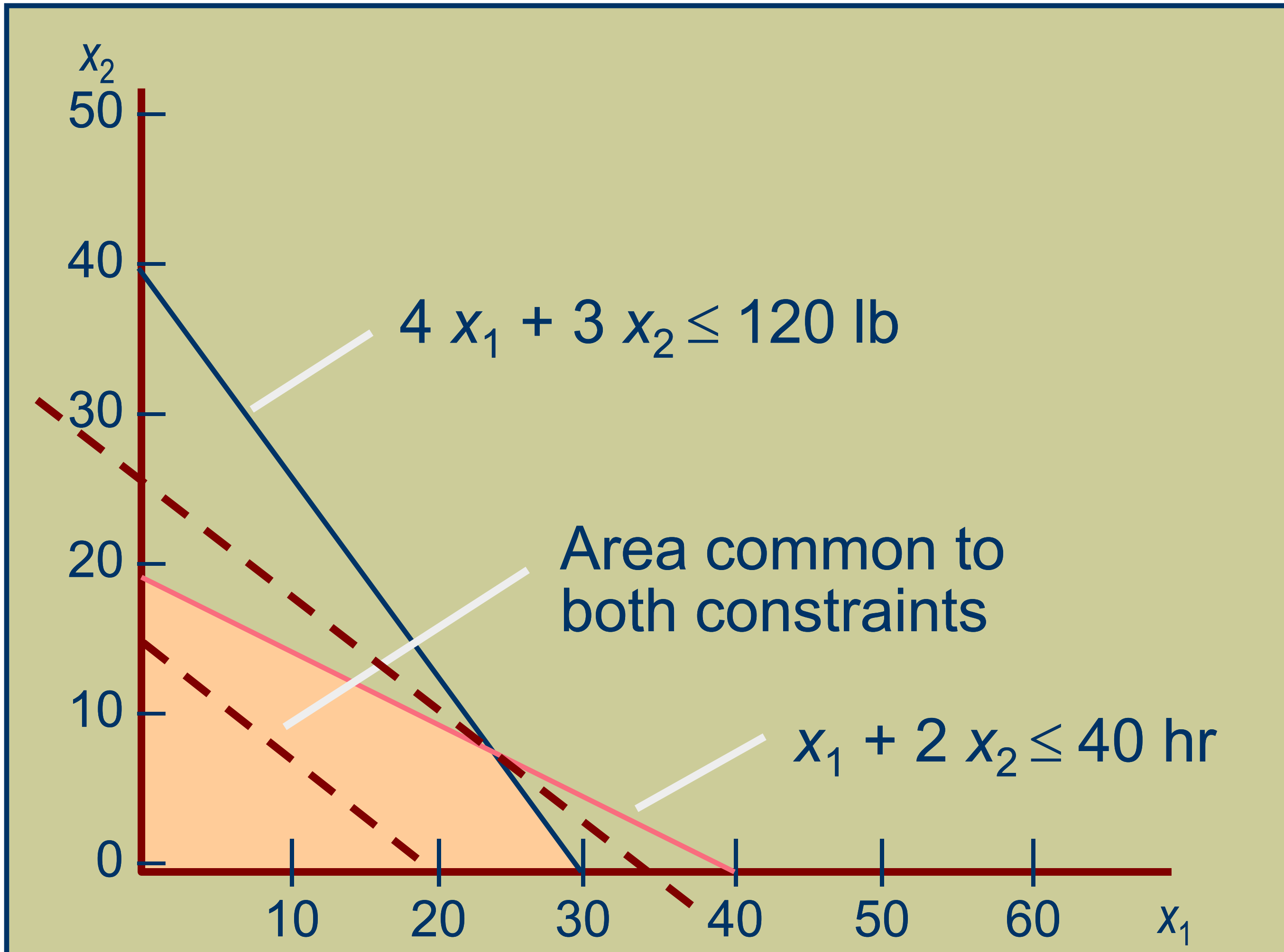
Solution is $x_1 = 24$ bowls $x_2 = 8$ mugs

Revenue = $1,360

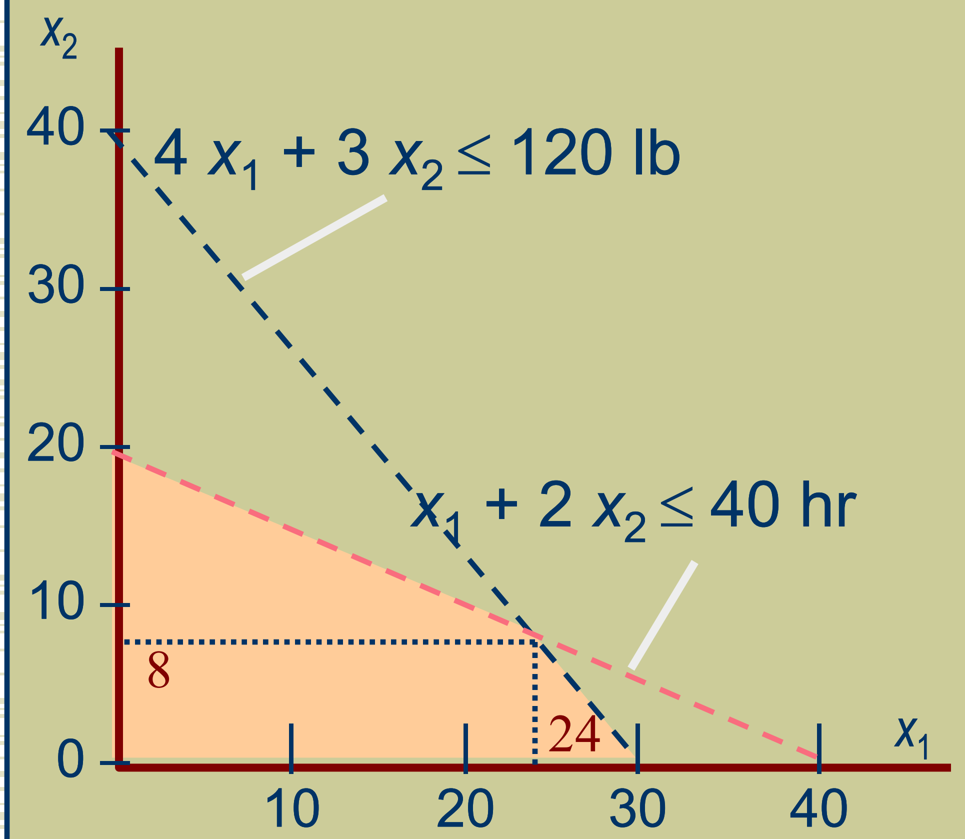
# Graphical Solution Method

1. Plot model constraint on a set of coordinates in a plane
2. Identify the feasible solution space on the graph where all constraints are satisfied simultaneously
3. Plot objective function to find the point on boundary of this space that maximizes (or minimizes) value of objective function

# Graphical Solution: Example



# Computing Optimal Values




$$x_1 + 2x_2 = 40$$
$$4x_1 + 3x_2 = 120$$

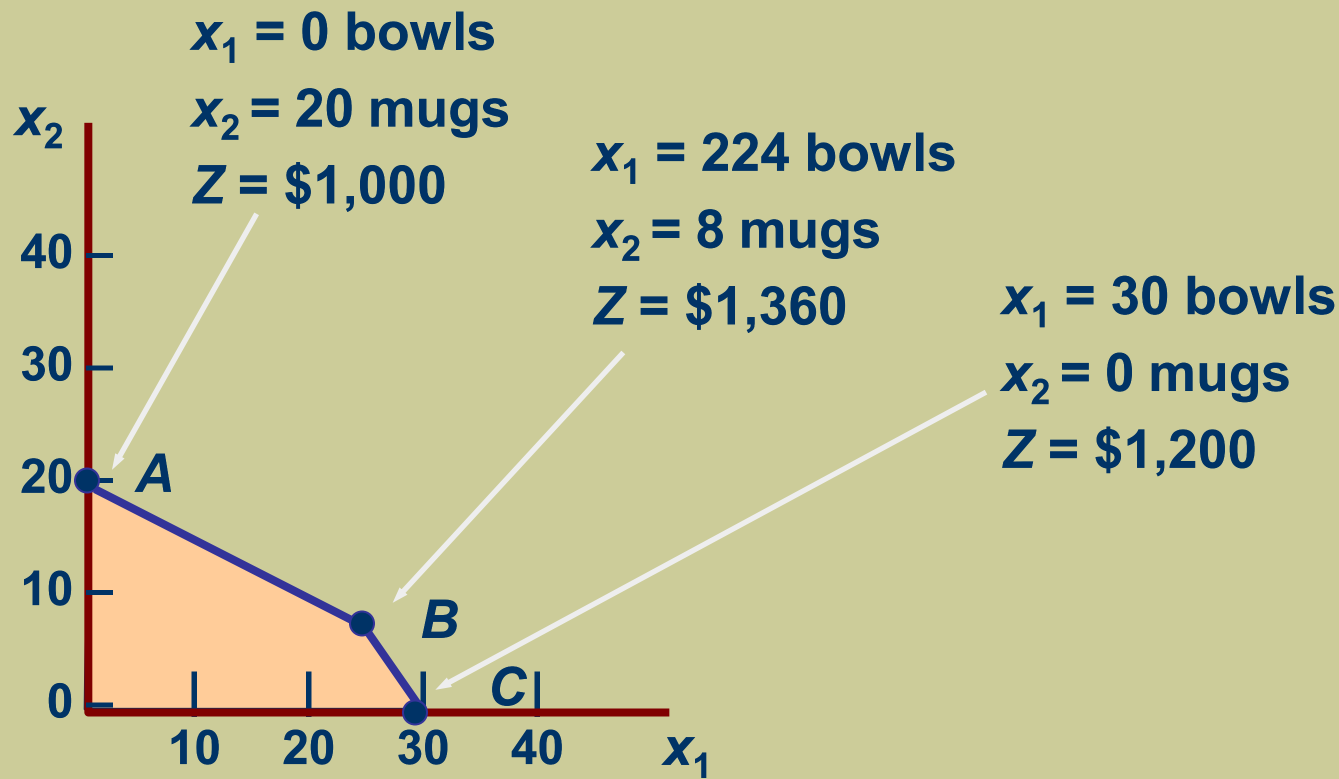
$$4x_1 + 8x_2 = 160$$
$$-4x_1 - 3x_2 = -120$$

$$5x_2 = 40$$
$$x_2 = 8$$

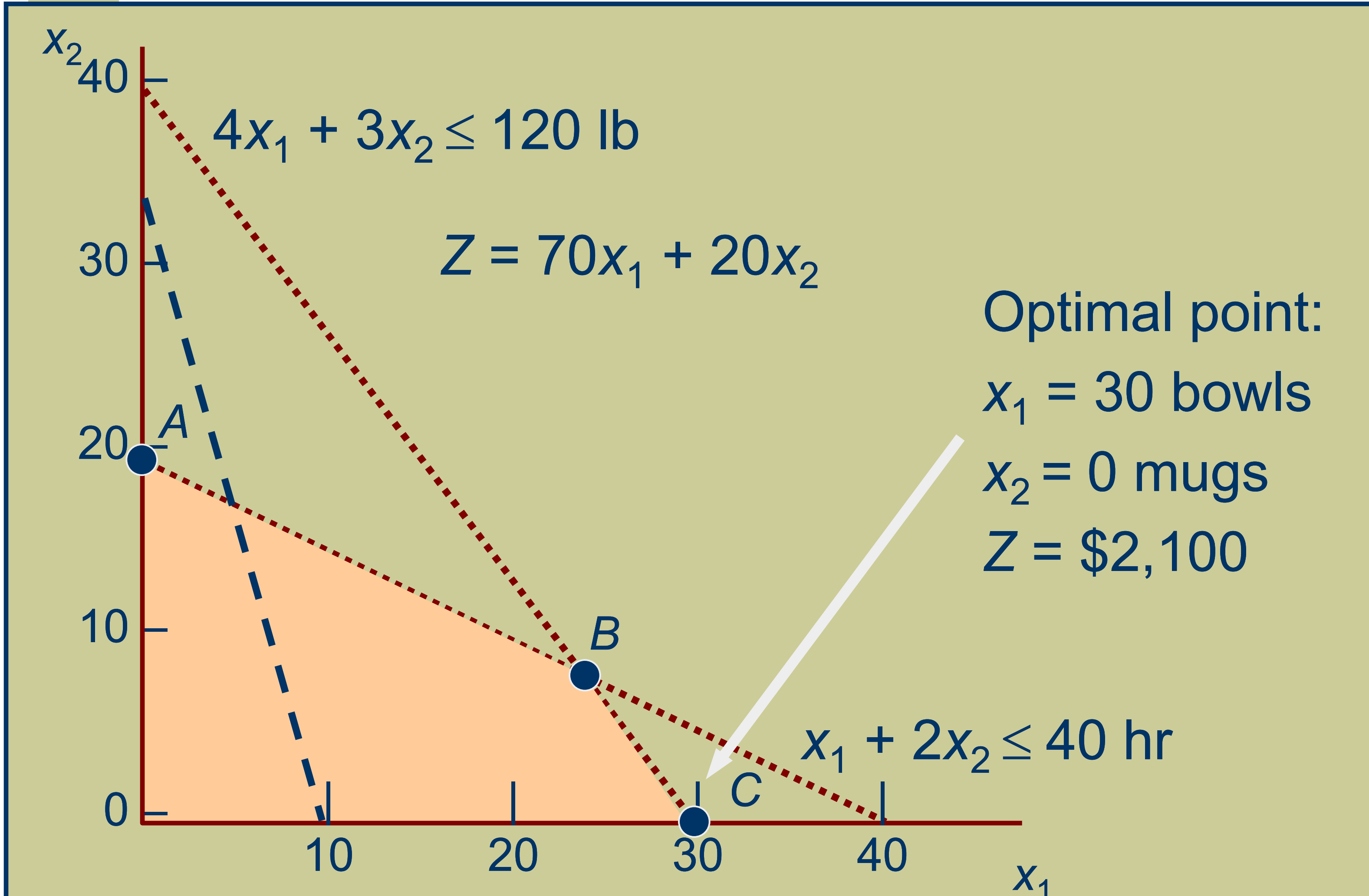
$$x_1 + 2(8) = 40$$
$$x_1 = 24$$

$$Z = \$50(24) + \$50(8) = \$1{,}360$$

# Extreme Corner Points



# Objective Function



# Minimization Problem

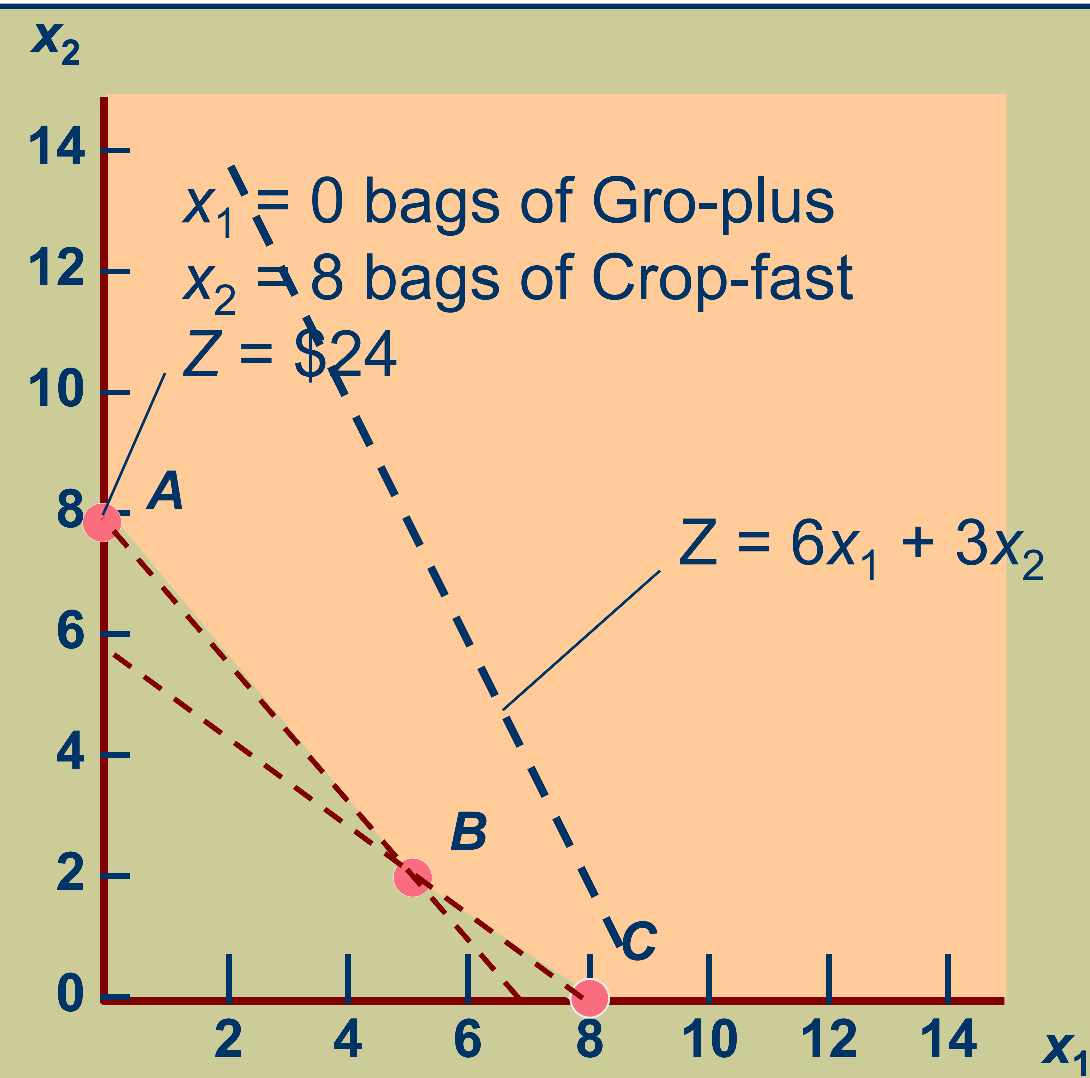
| | CHEMICAL CONTRIBUTION | |
|---|---|---|
| *Brand* | *Nitrogen (lb/bag)* | *Phosphate (lb/bag)* |
| Gro-plus | 2 | 4 |
| Crop-fast | 4 | 3 |

Minimize $Z = \$6x_1 + \$3x_2$

subject to

$$2x_1 + 4x_2 \geq 16 \text{ lb of nitrogen}$$
$$4x_1 + 3x_2 \geq 24 \text{ lb of phosphate}$$
$$x_1, x_2 \geq 0$$

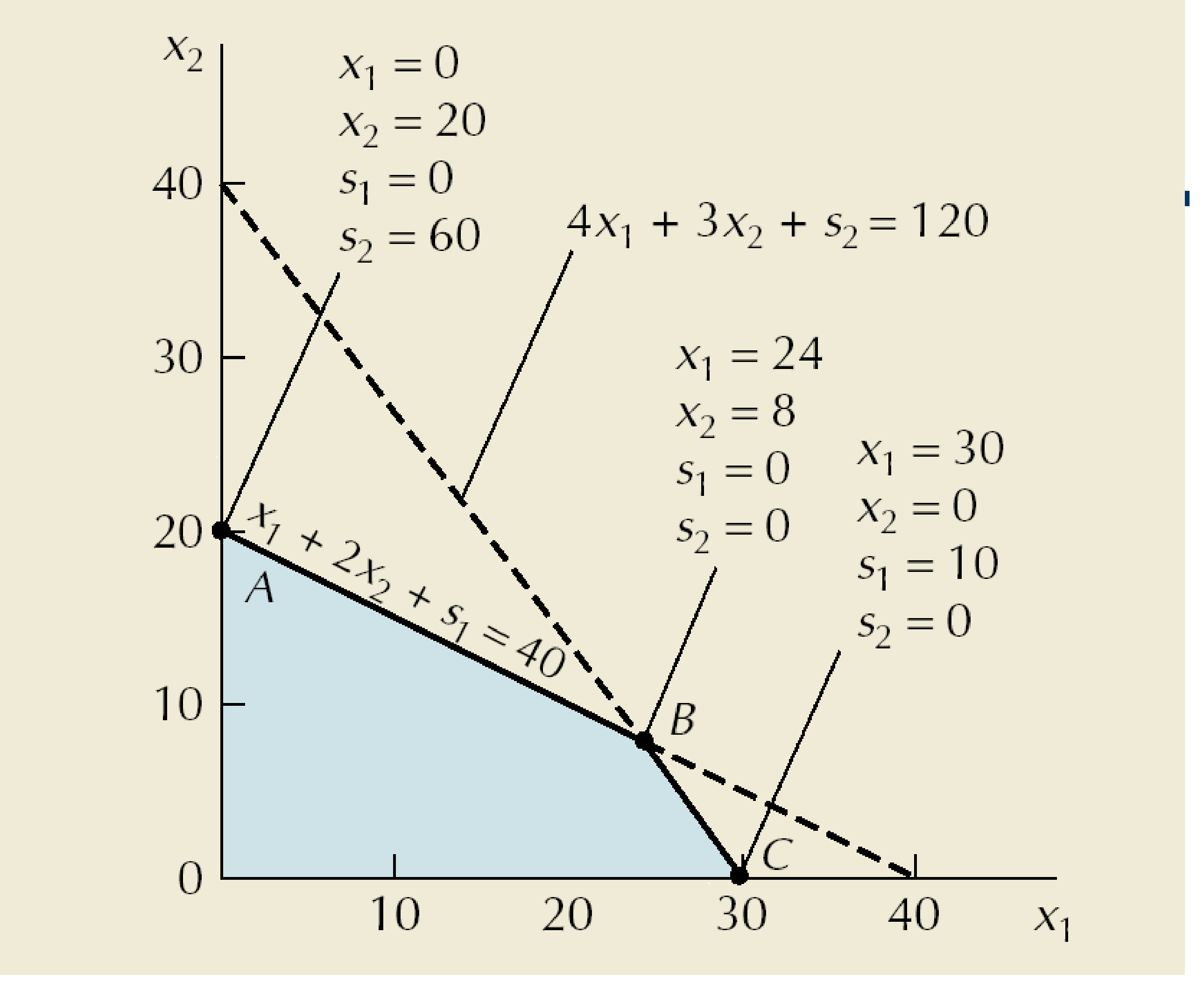
# Graphical Solution



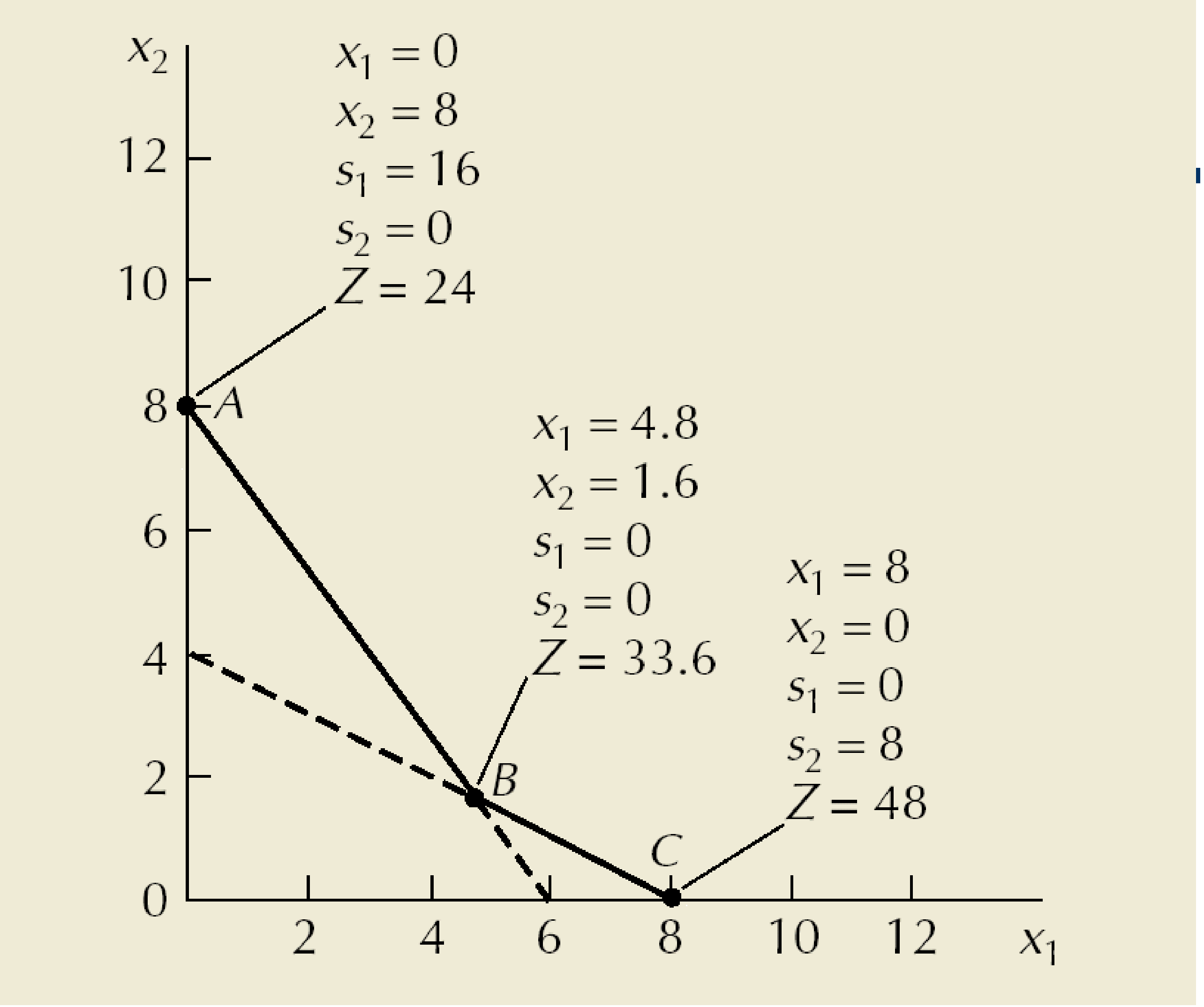
# Simplex Method

- A mathematical procedure for solving linear programming problems according to a set of steps
- Slack variables added to ≤ constraints to represent unused resources
  - $x_1 + 2x_2 + s_1 =$ □40 hours of labor
  - $4x_1 + 3x_2 + s_2 =$ □120 lb of clay
- Surplus variables subtracted from ≥ constraints to represent excess above resource requirement. For example
  - $2x_1 + 4x_2 \geq$ □16 is transformed into
  - $2x_1 + 4x_2 - s_1 =$ □16
- Slack/surplus variables have a 0 coefficient in the objective function
  - $Z = \$40x_1 + \$50x_2 + 0s_1 + 0s_2$

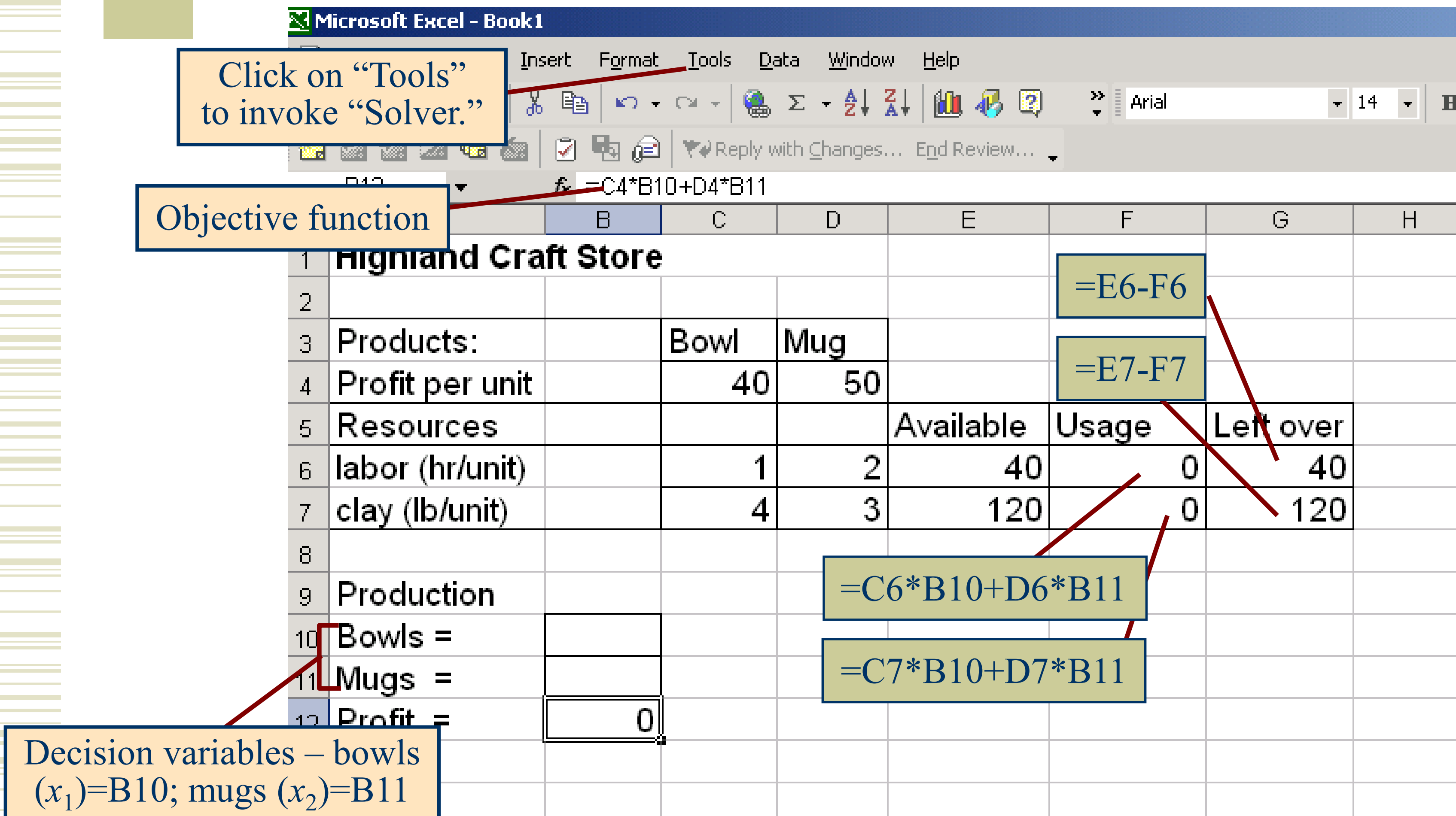
# Solution Points with Slack Variables



# Solution Points with Surplus Variables



# Solving LP Problems with Excel



# Solving LP Problems with Excel (cont.)

After all parameters and constraints have been input, click on “Solve.”

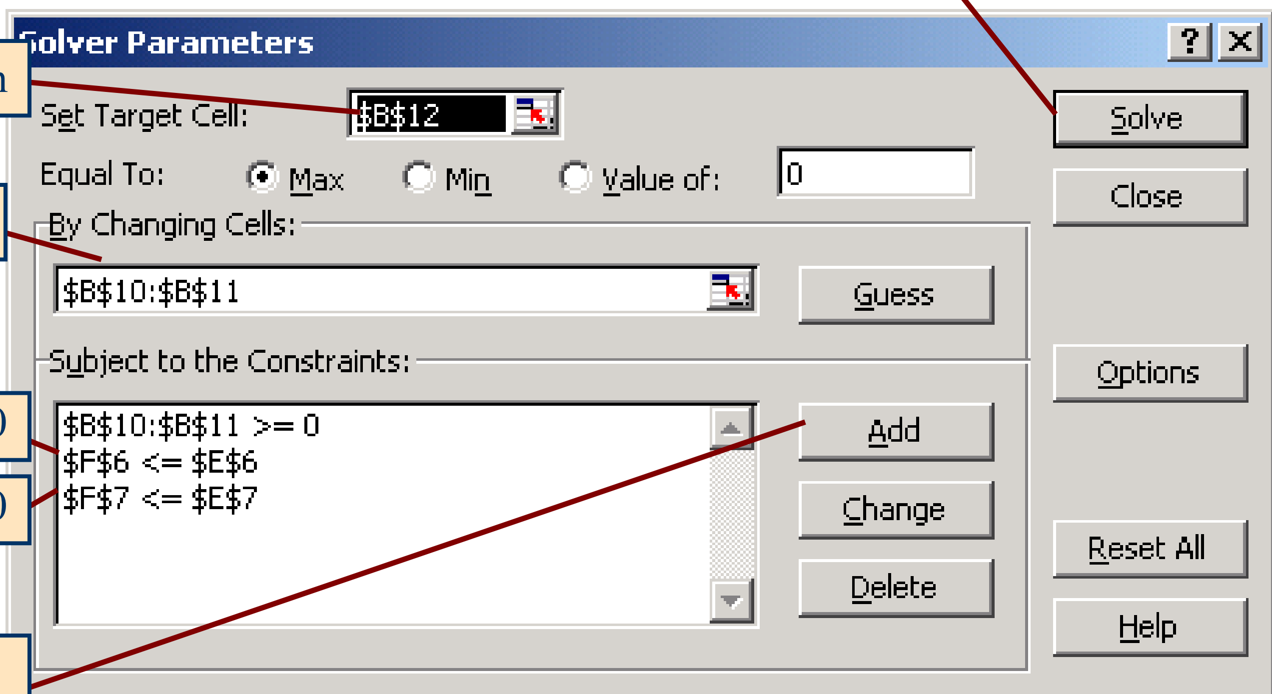
Objective function

Decision variables

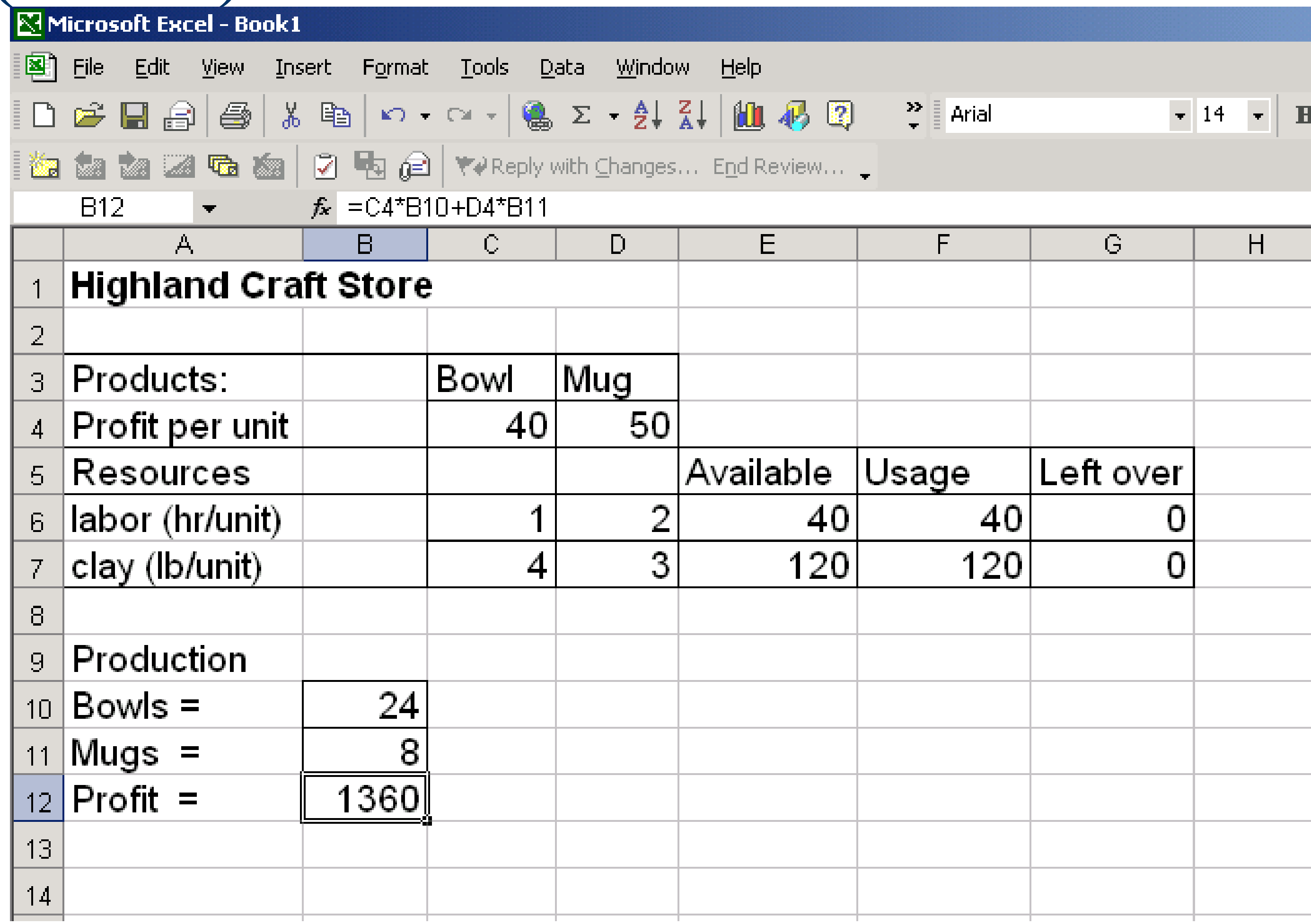
C6*B10+D6*B11≤40

C7*B10+D7*B11≤120

Click on “Add” to insert constraints

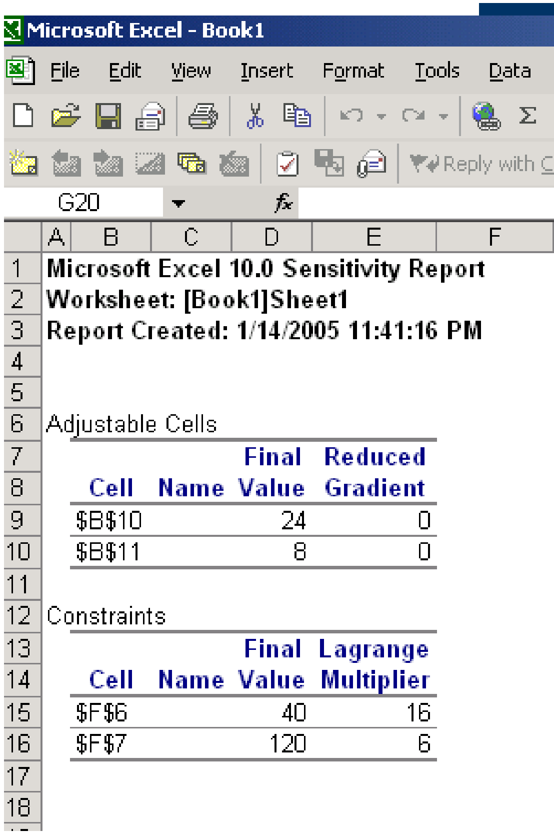


# Solving LP Problems with Excel (cont.)




| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Highland Craft Store** | | | | | | | |
| 2 | | | | | | | | |
| 3 | Products: | | Bowl | Mug | | | | |
| 4 | Profit per unit | | 40 | 50 | | | | |
| 5 | Resources | | | | Available | Usage | Left over | |
| 6 | labor (hr/unit) | | 1 | 2 | 40 | 40 | 0 | |
| 7 | clay (lb/unit) | | 4 | 3 | 120 | 120 | 0 | |
| 8 | | | | | | | | |
| 9 | Production | | | | | | | |
| 10 | Bowls = | 24 | | | | | | |
| 11 | Mugs = | 8 | | | | | | |
| 12 | Profit = | 1360 | | | | | | |
| 13 | | | | | | | | |
| 14 | | | | | | | | |

# Sensitivity Analysis



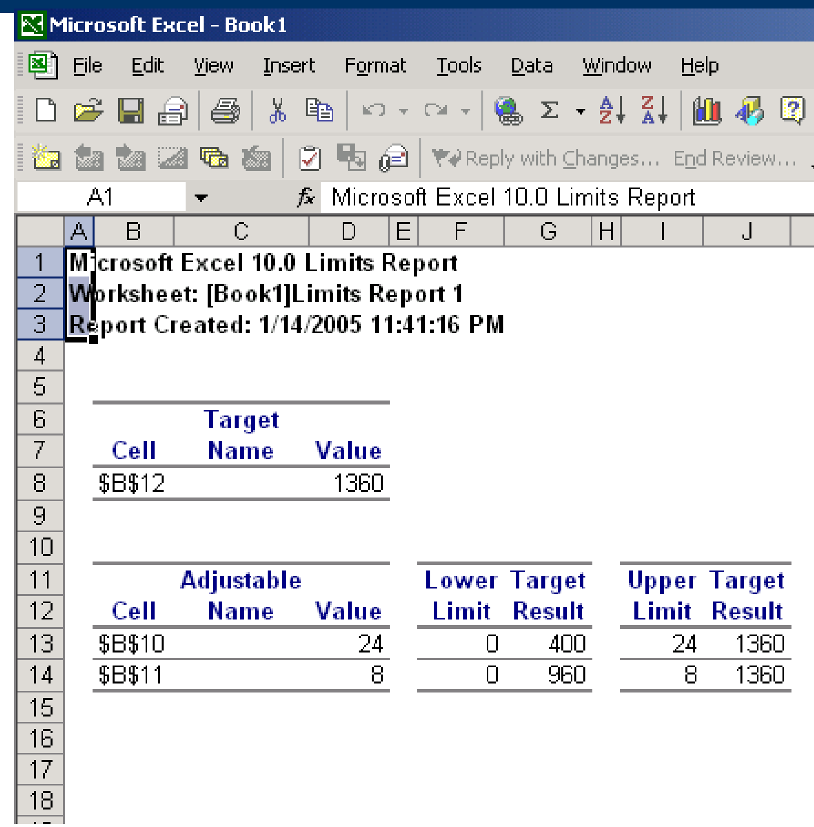
Microsoft Excel 10.0 Sensitivity Report
Worksheet: [Book1]Sheet1
Report Created: 1/14/2005 11:41:16 PM

Adjustable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $B$10 | | 24 | 0 |
| $B$11 | | 8 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $F$6 | | 40 | 16 |
| $F$7 | | 120 | 6 |

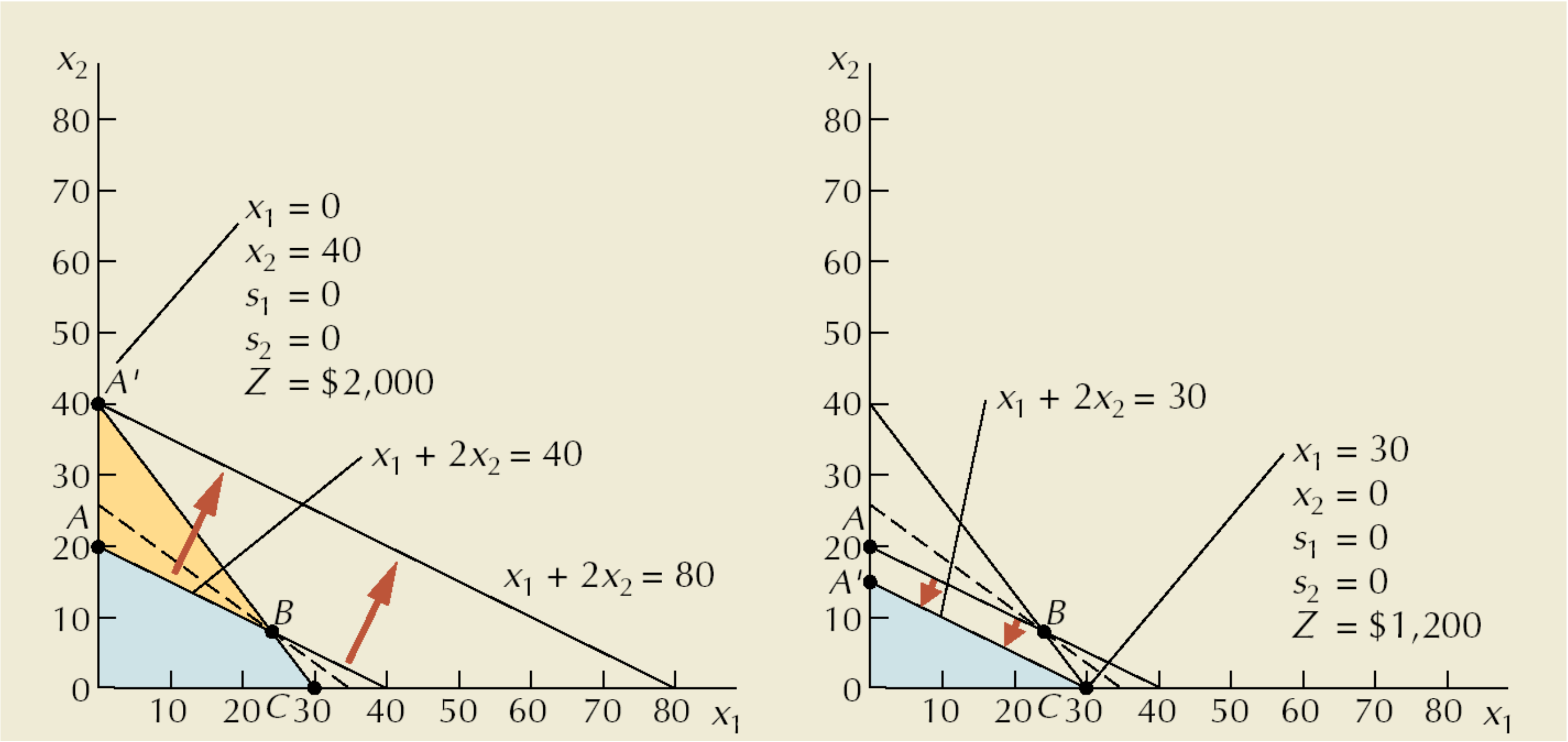


Microsoft Excel 10.0 Limits Report
Worksheet: [Book1]Limits Report 1
Report Created: 1/14/2005 11:41:16 PM

| Cell | Target Name | Value |
|---|---|---|
| $B$12 | | 1360 |

| Cell | Adjustable Name | Value | Lower Limit | Target Result | Upper Limit | Target Result |
|---|---|---|---|---|---|---|
| $B$10 | | 24 | 0 | 400 | 24 | 1360 |
| $B$11 | | 8 | 0 | 960 | 8 | 1360 |

# Sensitivity Range for Labor Hours



# Sensitivity Range for Bowls