Simplex Problems

Chapter Topics

- Simplex Method
- Characteristics of Simplex Method
- Why we should study the Simplex Method?
- Summary of the Simplex Method
- Examples solved by conducting tabular method

SIMPLEX METHOD

- Linear programming models could be solved algebraically. The most widely used algebraic procedure for solving linear programming problem is called the Simplex Method. The simplex method is a general-purpose linear-programming algorithm widely used to solve large scale problems. Although it lacks the intuitive appeal of the graphical approach, its ability to handle problems with more than two decision variables makes it extremely valuable for solving problems often encountered in production/operations management. Thus simplex method offers an efficient means of solving more complex linear programming problems.

Characteristics of Simplex Method

- In the simplex method, the computational routine is an iterative process. To *iterate* means to repeat; hence, in working toward the optimum solution, the computational routine is repeated over and over, following a standard pattern.
- Successive solutions are developed in a systematic pattern until the best solution is reached.
- Each new solution will yield a value of the objective function as large as or larger than the previous solution. This important feature assures us that we are always moving closer to the optimum answer. Finally, the method indicates when the optimum solution has been reached.

Characteristics of Simplex Method

- Most real-life linear programming problems have more than two variables, so a procedure called the simplex method is used to solve such problems. This procedure solves the problem in an iterative manner, that is, repeating the same set of procedures time after time until an optimal solution is reached. Each iteration brings a higher value for the objective function so that we are always moving closer to the optimal solution.

Characteristics of Simplex Method

The simplex method requires simple mathematical operations (addition, subtraction, multiplication, and division), but the computations are lengthy and tedious, and the slightest error can lead to a good deal of frustration. For these reasons, most users of the technique rely on computers to handle the computations while they concentrate on the solutions. Still, some familiarity with manual computations is helpful in understanding the simplex process. The student will discover that it is better not to use his/her calculator in working through these problems because rounding can easily distort the results. Instead, it is better to work with numbers in fractional form.

Why we should study the Simplex Method?

- It is important to understand the ideas used to produce solution. The simplex approach yields not only the optimal solution to the xi variables, and the maximum profit (or minimum cost) but valuable economic information as well.

- To be able to use computers successfully and to interpret LP computer print outs, we need to know what the simplex method is doing and why.

- We begin by solving a maximization problem using the simplex method. We then tackle a minimization problem.

SUMMARY OF THE SIMPLEX METHOD

Step 1. Formulate a LP model of the problem.

Step 2. Add slack variables to each constraint to obtain standard form.

Step 3. Set up the initial simplex tableau.

Step 4. Choose the nonbasic variable with the largest entry in the net evaluation row ($C_j - Z_j$) to bring into the basis. This identifies the pivot (key) column; the column associated with the incoming variable.

Step 5. Choose as the pivot row that row with the smallest ratio of "b_i / a_{ij}", for $a_{ij} > 0$ where j is the pivot column. This identifies the pivot row, the row of the variable leaving the basis when variable j enters.

Step 6. a). Divide each element of the pivot row by the pivot element.

b). According to the entering variable, find the new values for remaining variables.

Step 7. Test for optimality. If $C_j - Z_j \le 0$ for all columns, we have the optimal solution. If not, return to step 4.

Example 1

A Furniture Ltd., wants to determine the most profitable combination of products to manufacture given that its resources are limited. The Furniture Ltd., makes two products, *tables* and *chairs*, which must be processed through *assembly* and *finishing departments*. Assembly has 60 hours available; Finishing can handle up to 48 hours of work.

Manufacturing one table requires 4 hours in assembly and 2 hours in finishing.

Each chair requires 2 hours in assembly and 4 hours in finishing. Profit is $8 per table and $6 per chair.

	Hours required for 1 unit of product		Total hours
	Tables	*Chairs*	available
Assembly	4	2	60
Finishing	2	4	48
Profit per unit	$8	$6	

Tabular solution for Example 1/1

Stated algebraically, the Ltd., problem is

Maximise! Profit $Z = 8X_1 + 6X_2$

Subject to :

Assembly	$4X_1 + 2X_2 \le 60$
Finishing	$2X_1 + 4X_2 \le 48$
	All variables ≥ 0

- The first step is to convert the inequalities into equations.

The best combination of tables and chairs may not necessarily use all the time available in each department. We must therefore add to each inequality a variable, which will take up the slack, i.e. the time not used in each department. This variable is called a slack variable.

By adding the slack variables we convert the constraint inequalities in the problem into equations. The slack variable in each department takes on whatever value is required to make the equation relationship hold.

$\therefore$ The final form is

Maximize Profit $Z = 8X_1 + 6X_2 + 0S_1 + 0S_2$

Subject to

$$4X_1 + 2X_2 + S_1 = 60$$

$$2X_1 + 4X_2 + S_2 = 60$$

All variables ≥ 0

Tabular solution for Example 1/2

- The 2nd step is to put the equations into tabular form, called ***tableaus***.

Product mix column

Profit per unit column   constant column (quantities of product in the mix)

Variable columns

Cj			$8	6	0	0	Cj row
	Product mix	Quantity	X_1	X_2	S_1	S_2	Variable row
$0	S_1	60	4	2	1	0	
0	S_2	48	2	4	0	1	

Real products   slack time

The simplest starting solution is to make no tables or chairs, have all unused time and earn no profit. This solution is technically feasible but not financially attractive. (Because the variables X_1 and X_2 do not appear in the mix, they are equal to zero.)

To find the profit for each solution and to determine whether the solution can be improved upon, we need to add two more rows to the initial simplex tableau: a Z_j row and a $C_j - Z_j$ row.

Column Z_j = Total profit from this particular solution

Tabular solution for Example 1/3

The four values for Z_j under the variable columns (all 0 $) are the amounts by which profit would be reduced if 1 unit of any of the variables were added to the mix.

C_j			$8	6	0	0	
	Product mix	Quantity	X_1	X_2	S_1	S_2	
$0	S_1	60	4	2	1	0	
0	S_2	48	2	4	0	1	
	Z_j	$0	0	0	0	0	
	$C_j - Z_j$		8	6	0	0	

Max

Z_j represents the gross profit given up by adding 1 unit of this variable into the current solution (profit loss per unit). $C_j - Z_j$ is net profit from the introduction 1 unit of each variable into the solution.

By examining the numbers in the $C_j - Z_j$ row we can see that total profit can be increased by 48 for each unit of X_1 (tables). Positive number indicates that profits can be improved for each unit added. We select the largest positive value. Max $C_j - Z_j$ value showing the variable that should be added, replacing one of the variables present in the mix.

Tabular solution for Example 1/4

- The next step is to determine which variable will be replaced.

  This is done in the following manner:

  Divide quantity column values by their corresponding numbers in the maximum (optimum) column and select the row with the smallest nonnegative ratio as the row to be replaced.

  S_1 row 60/4 = 15 units of Table (X_1) → minimum replaced row.

  S_2 row 48/4 = 24 units of Table (X_1)

1st Simplex Tableau

C_j			\$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	b_i/a_{ij}
0	S_1	60	4	2	1	0	60/4 = 15
0	S_2	48	2	4	0	1	48/2 = 24
	Z_j	\$0	\$0	0	0	0	Intersectional elements
	$C_j - Z_j$		8	6	0	0	(key #)

↑ Max. (optimum entering variable)

Tabular solution for Example 1/5

a_{ij} = coefficient associated with variable j in the constraint i

For 2nd Simplex Tableau

X_1 = 60/4 = 15, 4/4 = 1, 2/4 = 1/2, 1/4 = 1/4 , 0/4 = 0

Thus new X_1 row should be (15, 1 , 1/2 , 1/4 , 0),

The new values for remaining rows:

[elements in old row] - [key #] x [corresponding elements in replacing row] = new row

Elements in old row	–	key #	x	replacing row	=	new row
48	–	2	x	15	=	18
2	–	2	x	1	=	0
4	–	2	x	1/2	=	3
0	–	2	x	1/4	=	-1/2
1	–	2	x	0	=	1

The computation of Z_j row for 2nd tableau is as follows.

Z_j for X_1 8 x 1 + 0 (0) = 8
Z_j for X_2 8 (1/2) + 0 (3) = 4
Z_j for S_1 8 (1/4) + 0 (-1/2) = 2
Z_j for S_2 8 (0) + 0 (1) = 0

} Profit given up by introducing 1 unit of these variables

Z_j (total profit) = 8 (15) + 0 (18) = $120

Tabular solution for Example 1/6

2nd Simplex Tableau

C_j			$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	b_i/a_{ij}
$8	X_1	15	1	1/2	1/4	0	15/ ½ = 30
0	S_2	18	0	3	-1/2	1	18/3 = 6 → min! leaving
	Z_j	$120	$8	4	2	0	
	$C_j - Z_j$		$0	2	-2	0	

↑ Max! entering

X_2 will enter in the product mix and S_2 is leaving.

New X_2 values: 18/3 = 6, 0/3 = 0, 3/3 = 1, -½ /3 = -1/6, 1/3 = 1/3

Thus new X_2 (replacing row) values = 6, 0, 1, -1/6, 1/3 (Assumes same row position as the replaced row)

Tabular solution for Example 1/7

New Values for X_1 :

Elements in old X_1 row	–	key #	x	replacing row	=	new X_1 row
15	–	1/2	x	6	=	12
1	–	1/2	x	0	=	1
1/2	–	1/2	x	1	=	0
1/4	–	1/2	x	-1/6	=	1/3
0	–	1/2	x	1/3	=	-1/6

New Z_j values :

Z_j (total profit) = 8 (12) + 6 (6) = $132

Z_j for X_1 8 (1) + 6 (0) = 8

Z_j for X_2 8 (0) + 6 (1) = 6

Z_j for S_1 8 (1/3) + 6 (-1/6) = 5/3

Z_j for S_2 8 (- 1/6) + 6 (1/3) = 2/3

Tabular solution for Example 1/8

3rd Simplex Tableau

C_j			\$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	
\$8	X_1	12	1	0	1/3	-1/6	
6	X_2	6	0	1	-1/6	1/3	
	Z_j	\$132	\$8	6	5/3	2/3	
	$C_j - Z_j$		\$0	0	-5/3	-2/3	

There is no positive "$C_j - Z_j$" value, no further profit improvement is possible. Thus the optimum solution is obtained. Profit will be maximized by making 12 tables and 6 chairs and having no unused time in either department (because slack variables do no appear in the product-mix column and are equal to zero). Optimum profit is \$132 .

Verification:

Objective function

$$Z_j = 8\,X_1 + 6\,X_2 + 0\;(S_1 + S_2)$$
$$Z_j = 8\,(12) + 6\,(6) + 0 = \$132$$

Constraints :

Assembly $4\,X_1 + 2\,X_2 \leq 60 \rightarrow 4\,(12) + 2\,(6) \leq 60 \rightarrow 60 \leq 60$

Finishing $2\,X_1 + 4\,X_2 \leq 48 \rightarrow 2\,(12) + 4\,(6) \leq 48 \rightarrow 48 \leq 48$

Example 2

PAR Inc. produces golf equipment and decided to move into the market for standard and deluxe golf bags. Each golf bag requires the following operations:

Cutting and dyeing the material,

Sewing,

Finishing (inserting umbrella holder, club separators etc.),

Inspection and packaging.

Each standard golf-bag will require 7/10 hr. in the cutting and dyeing department, 1/2 hr. in the sewing department, 1 hr. in the finishing department and 1/10 hr. in the inspection & packaging department.

Deluxe model will require 1 hr. in the cutting and dyeing department, 5/6 hr. for sewing, 2/3 hr. for finishing and 1/4 hr. for inspection and packaging

The profit contribution for every standard bag is 10 MU and for every deluxe bag is 9 MU.

In addition the total hours available during the next 3 months are as follows:

Cutting & dyeing dept	630 hrs
Sewing dept	600 hrs
Finishing	708 hrs
Inspection & packaging	135 hrs

The company's problem is to determine how many standard and deluxe bags should be produced in the next 3 months?

Example 2

Let X_1 = number of standard bags

X_2 = number of deluxe bags

Z = the total profit contribution

Objective function:

$$\text{Max! } Z = 10\,X_1 + 9\,X_2$$

Subject to constraints

$$\tfrac{7}{10}X_1 + 1\,X_2 \le 630 \qquad \text{cutting and dying}$$

$$\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 \le 600 \qquad \text{sewing}$$

$$1\,X_1 + \tfrac{2}{3}X_2 \le 708 \qquad \text{finishing}$$

$$\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 \le 135 \qquad \text{inspection \& packaging}$$

$$\left.\begin{array}{l} X_1 \ge 0 \\ X_1 \ge 0 \end{array}\right\} \text{ Nonnegative constraints}$$

In linear programming terminology, any unused or idle capacity for a ≤ constraint is referred to as the slack associated with the constraint. Often variables, called slack variables, are added to the formulation of a linear programming problem to represent the slack or idle capacity. Unused capacity makes no contribution to profit; thus slack variables have coefficients of zero in the objective function. Whenever a linear program is written in a form with all constraints expressed as equalities, it is said to be written in standard form.

After the addition of slack variables to the mathematical statement, the mathematical model becomes

$$\text{Max! } \quad 10\,X_1 + 9\,X_2 + 0\,S_1 + 0\,S_2 + 0\,S_3 + 0\,S_4$$

Subject to

$$\tfrac{7}{10}X_1 + 1\,X_2 + 1\,S_1 = 630$$

$$\tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 + 1\,S_2 = 600$$

$$1\,X_1 + \tfrac{2}{3}X_2 + 1\,S_3 = 708$$

$$\tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 + 1\,S_4 = 135$$

$$X_1, X_2, S_1, S_2, S_3, S_4 \ge 0$$

Tabular solution for Example 2/1

Initial Tableau

C_j			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	b_i / a_{ij}
0 MU	S_1	630	7/10	1	1	0	0	0	630/ 7/10 =900
0 MU	S_2	600	½	5/6	0	1	0	0	600/ ½ = 1200
0 MU	S_3	708	1	2/3	0	0	1	0	708/1 = 708 (min leaving)
0 MU	S_4	135	1/10	¼	0	0	0	1	135/ 1/10 = 1350
	Z_j	0 MU	0 MU	0 MU	0 MU	0 MU	0 MU	0 MU	
	$C_j - Z_j$		10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	

Max. (Entering)

For the 2nd Simplex Tableau

708/1 = 708, 1/1 = 1, 2/3/1 = 2/3, 0, 0, 1, 0

∴ new X_1 value are : 708, 1, 2/3, 0, 0, 1, 0

Tabular solution for Example 2/2

Elements in old S_1 row	–	key #	x	new X_1 row	=	new S_1 row
630	–	7/10	x	708	=	134.4
7/10	–	7/10	x	1	=	0
1	–	7/10	x	2/3	=	8/15
1	–	7/10	x	0	=	1
0	–	7/10	x	0	=	0
0	–	7/10	x	1	=	-7/10
0	–	7/10	x	0	=	0

Elements in old S_2 row	–	key #	x	new X_1 row	=	new S_2 row
600	–	1/2	x	708	=	246
1/2	–	1/2	x	1	=	0
5/6	–	1/2	x	2/3	=	1/2
0	–	1/2	x	0	=	0
1	–	1/2	x	0	=	1
0	–	1/2	x	1	=	-1/2
0	–	1/2	x	0	=	0

Tabular solution for Example 2/3

Elements in old S_4 row	–	key #	x	new X_1 row	=	new S_4 row
135	–	1/10	x	708	=	64.2
1/10	–	1/10	x	1	=	0
1/4	–	1/10	x	2/3	=	11/60
0	–	1/10	x	0	=	0
0	–	1/10	x	0	=	0
0	–	1/10	x	1	=	-1/10
1	–	1/10	x	0	=	1

2^{nd} Tableau

C_j			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	b_i / a_{ij}
0 MU	S_1	134.4	0	8/15	1	0	-7/10	0	252 (min leaving)
0 MU	S_2	246	0	½	0	1	-1/2	0	492
10 MU	X_1	708	1	2/3	0	0	1	0	1062
0 MU	S_4	64.2	0	11/60	0	0	-1/10	1	3852/11
	Z_j	7080 MU	10	20/3	0	0	10	0	
	$C_j - Z_j$		0	7/3	0	0	-10	0	

Max. (Entering)

New X_2 values are: $\frac{134.4}{\ }$ = 252 , 0, 1, 15/8, 0, -21/16, 0

Tabular solution for Example 2/4

Elements in old S_2 row	–	key #	x	new X_2 row	=	new S_2 row
246	–	1/2	x	252	=	120
0	–	1/2	x	0	=	0
1/2	–	1/2	x	1	=	0
0	–	1/2	x	15/8	=	-15/16
1	–	1/2	x	0	=	1
-1/2	–	1/2	x	-21/16	=	5/32
0	–	1/2	x	0	=	0

Elements in old X_1 row	–	key #	x	new X_2 row	=	new X_1 row
708	–	2/3	x	252	=	540
1	–	2/3	x	0	=	1
2/3	–	2/3	x	1	=	0
0	–	2/3	x	15/8	=	-5/4
0	–	2/3	x	0	=	0
1	–	2/3	x	-21/16	=	15/8
0	–	2/3	x	0	=	0

Elements in old S_4 row	–	key #	x	new X_2 row	=	new S_4 row
64.2	–	11/60	x	252	=	18
0	–	11/60	x	0	=	0
11/60	–	11/60	x	1	=	0
0	–	11/60	x	15/8	=	-11/32
0	–	11/60	x	0	=	0
-1/10	–	11/60	x	-21/10	=	45/320
1	–	11/60	x	0	=	1

Tabular solution for Example 2/5

3rd Tableau

Cj			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4
9 MU	X_2	252	0	1	15/8	0	-21/16	0
0 MU	S_2	120	0	0	-15/16	1	5/32	0
10 MU	X_1	540	1	0	-5/4	0	15/8	0
0 MU	S_4	18	0	0	-11/32	0	45/320	1
	Zj	7668 MU	10	9	135/8+(5/4) = 35/8	0	-189/16+ 300/16=111/16	0
	Cj – Zj		0	0	-35/8	0	-111/16	0

There is no positive Cj – Zj value in the simplex tableau. Therefore no further profit improvement is possible. Thus the optimum solution is obtained.

- Thus: Standard bag production (X_1) = 540 bags .
  Deluxe bag production (X_2) = 252 bags .
- Maximum profit = Z = \$10 (540) + \$9 (252) = \$7668
- Unused hours in Sewing department = 120 hours
  Inspection and packaging department = 18 hours

Example 3

High Tech industries import components for production of two different models of personal computers, called deskpro and portable. High Tech's management is currently interested in developing a weekly production schedule for both products.

The deskpro generates a profit contribution of $50/unit, and portable generates a profit contribution of $40/unit. For next week's production, a max of 150 hours of assembly time is available. Each unit of deskpro requires 3 hours of assembly time. And each unit of portable requires 5 hours of assembly time.

High Tech currently has only 20 portable display components in inventory; thus no more than 20 units of portable may be assembled. Only 300 sq. feet of warehouse space can be made available for new production. Assembly of each Deskpro requires 8 sq. ft. of warehouse space, and each Portable requires 5 sq. ft. of warehouse space.

	X_1 - Deskpro	X_2 – Portable	Capacity
Assembly line	3	5	150
Portable Ass	-	1	20
Space	8	5	300
Profit Cont.	$50/unit	$40/unit	

Tabular solution for Example 3/1

	X_1 - Deskpro	X_2 – Portable	Capacity
Assembly line	3	5	150
Portable Ass	-	1	20
Space	8	5	300
Profit Cont.	$50/unit	$40/unit	

X_1 = *number of units of the Deskpro* X_2 = *number of units of the Portable*

Objective Function : Max! $Z = 50 X_1 + 40 X_2$

Subject to :

$3 X_1 + 5 X_2 \leq 150$ Assembly time

$1 X_2 \leq 20$ Portable display

$8 X_1 + 5 X_2 \leq 300$ Warehouse capacity

$X_1, X_2 \geq 0$

Adding a slack variable to each of the constraints permits us to write the problem in standard form:

Objective Function: Max! $Z = 50 X_1 + 40 X_2 + 0 S_1 + 0S_2 + 0S_3$

Subject to:

$$3 X_1 + 5 X_2 + 1S_1 = 150$$

$$1 X_2 + 1S_2 = 20$$

$$8 X_1 + 5 X_2 + 1S_3 = 300$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Tabular solution for Example 3/2

Initial Tableau

C_j			\$50	\$40	\$0	\$0	\$0	
	Product mix	Quantity bi	X_1	X_2	S_1	S_2	S_3	b_i / a_{ij}
\$0	S_1	150	3	5	1	0	0	150/3 = 50
\$0	S_2	20	0	1	0	1	0	--
\$0	S_3	300	8	5	0	0	1	300/8 = 37.5 (min. leaving)
	Z_j	\$0	\$0	\$0	\$0	\$0	\$0	
	$C_j - Z_j$		\$50	\$40	\$0	\$0	\$0	

Max. (entering)

New X_1 value : 300/8 = 37.5, 8/8 = 1, 5/8, 0, 0, 1/8

Old S_1 row	–	key #	x	new X_1 row	=	new S_1 row
150	–	3	x	75/2	=	37.5
3	–	3	x	1	=	0
5	–	3	x	5/8	=	25/8
1	–	3	x	0	=	1
0	–	3	x	0	=	0
0	–	3	x	1/8	=	-3/8

Old S_2 row	–	key #	x	new X_1 row	=	new S_2 row
20	–	0	x	75/2	=	20
0	–	0	x	1	=	0
1	–	0	x	5/8	=	1
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	1/8	=	0

Tabular solution for Example 3/3

2nd Tableau

Cj			\$50	\$40	\$0	\$0	\$0	
	Product mix	Quantity bi	X_1	X_2	S_1	S_2	S_3	b_i / a_{ij}
\$0	S_1	75/2	0	25/8	1	0	-3/8	75/2 / 25/8 = 12 (min leaving)
\$0	S_2	20	0	1	0	1	0	20/1=20
\$50	X_1	75/2	1	5/8	0	0	1/8	75/2 / 5/8 = 60
	Zj	\$1875	\$50	\$250/8	\$0	\$0	\$50/8	
	Cj – Zj		\$0	\$70/8	\$0	\$0	\$-50/8	

Max. (Entering)

New X_2 values : 12, 0, 1, 8/25, 0, -3/25

old S_2 row	–	key #	x	new X_2 row	=	new S_2 row
20	–	1	x	12	=	8
0	–	1	x	0	=	0
1	–	1	x	1	=	0
0	–	1	x	8/25	=	-8/25
1	–	1	x	0	=	1
0	–	1	x	-3/25	=	3/25

old X_1 row	–	key #	x	new X_2 row	=	new X_1 row
75/2	–	5/8	x	12	=	30
1	–	5/8	x	0	=	1
5/8	–	5/8	x	1	=	0
0	–	5/8	x	8/25	=	-1/5
0	–	5/8	x	0	=	0
1/8	–	5/8	x	-3/25	=	1/5

Tabular solution for Example 3/3

3rd Tableau

C_j			\$50	\$40	\$0	\$0	\$0
	Product mix	Quantity Bi	X_1	X_2	S_1	S_2	S_3
\$40	X_2	12	0	1	8/25	0	-3/25
\$0	S_2	8	0	0	-8/25	1	3/25
\$50	X_1	30	1	0	-1/5	0	1/5
	Z_j	\$1980	\$50	\$40	\$14/5	\$0	\$26/5
	$C_j - Z_j$		\$0	\$0	\$-14/5	\$0	\$-26/5

The optimal solution to a linear programming problem has been reached when all of the entries in the net evaluation row $C_j - Z_j$ are zero or negative. In such cases, the optimal solution is the current basic feasible solution.

Thus:

Units of Deskpro production (X_1) = 30 units

Units of Portable production (X_2) = 12 units

S_2 = 8 units

Management should note that there would be eight unused Portable display units.

Maximum profit is \$1980.

Tableau Form : The Special Case

- Obtaining tableau form is somewhat more complex if the LP contains ≥ constraints, = constraints, and/or “-ve” right-hand-side values. Here we will explain how to develop tableau form for each of these situations.

Tabular solution for Example 4/1

- Suppose that in the high-tech industries problem, management wanted to ensure that the combined total production for both models would be at least 25 units.
- Thus,
- Objective Function Max $Z = 50X_1 + 40X_2$

  Subjective to : $3X_1 + 5X_2 \leq 150$ Assembly time
- $1X_1 \leq 20$ Portable display
- $8X_1 + 5X_2 \leq 300$ Warehouse space
- $1X_1 + 1X_2 \geq 25$ Min. total production
- $X_1, X_2 \geq 0$

Tabular solution for Example 4/2

- First, we use three slack variables and one surplus variable to write the problem in std. Form.
- Max $Z = 50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$
- Subject to $3X_1 + 5X_2 + 1S_1 = 150$
- $1X_2 + 1S_2 = 20$
- $8X_1 + 5X_2 + 1S_3 = 300$
- $1X_1 + 1X_2 - 1S_4 = 25$
- All variables ≥ 0
- For the initial tableau $X_1 = 0$ $X_2 = 0$
- $S_1 = 150$ $S_2 = 20$
- $S_3 = 300$ $S_4 = -25$

Tabular solution for Example 4/3

- Clearly this is not a basic feasible solution since

  $S4 = -25$ violates the nonnegativity requirement.

∴ We introduce new variable called ARTIFICIAL VARIABLE.

- Artificial variables will be eliminated before the optimal solution is reached. We assign a very large cost to the variable in the objective function.

∴ Objective function

$$50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 - MA_4$$

Tabular solution for Example 4/4

Initial Tableau

C_j			50	40	0	0	0	0	-M	
	Product mix	Quantit bi	X_1	X_2	S_1	S_2	S_3	S_4	A_4	b_i / a_{ij}
0	S_1	150	3	5	1	0	0	0	0	150/3=50
0	S_2	20	0	1	0	1	0	0	0	--
0	S_3	300	8	5	0	0	1	0	0	300/8=37.5
-M	A_4	25	1	1	0	0	0	-1	1	25 Min. leaving
	Z_j	-25M	-M	-M	0	0	0	M	-M	
	$C_j – Z_j$		50+M	40+M	0	0	0	-M	0	

Max. (Entering)

New X_1 values = 25, 1, 1, 0, 0, 0, -1, 1

Tabular solution for Example 4/5

Old S_1 row	–	key #	x	new X_1 values	=	new S_1 row
150	–	3	x	25	=	75
3	–	3	x	1	=	0
5	–	3	x	1	=	2
1	–	3	x	0	=	1
0	–	3	x	0	=	0
0	–	3	x	0	=	0
0	–	3	x	-1	=	3
0	–	3	x	1	=	-3

Old S_2 row	–	key #	x	new X_1 values	=	new S_2 row
20	–	0	x	25	=	20
0	–	0	x	1	=	0
1	–	0	x	1	=	1
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	-1	=	0
0	–	0	x	1	=	0

Old S_3 row	–	key #	x	new X_1 values	=	new S_3 row
300	–	8	x	25	=	10
8	–	8	x	1	=	0
5	–	8	x	1	=	-3
0	–	8	x	0	=	0
0	–	8	x	0	=	0
1	–	8	x	0	=	1
0	–	8	x	-1	=	8
0	–	8	x	1	=	-8

Tabular solution for Example 4/6

2nd Tableau

C_j	Prodt mix	Quant b_i	X_1	X_2	S_1	S_2	S_3	S_4	A_4	b_i / a_{ij}
			\$50	40	0	0	0	0	-M	
\$0	S_1	75	0	2	1	0	0	3	-3	75/3=25
0	S_2	20	0	1	0	1	0	0	0	--
0	S_3	100	0	-3	0	0	1	8	-8	100/8=12.5 Min,leaving
50	X_1	25	1	1	0	0	0	-1	1	--
	Z_j	\$1250	50	50	0	0	0	-50	50	
	$C_j – Z_j$		0	-10	0	0	0	50	-M-50	

new S4 values : 100/8 = 25/2, 0, -3/8, 0, 0, 1/8, 1

Max. (Entering)

Tabular solution for Example 4/7

- <u>IMPORTANT!!</u>
- Since A4 is an artificial variable that was added simply to obtain an initial basic feasible solution, we can drop its associated column from the simplex tableau.
- Indeed whenever artificial variables are used, they can be dropped from the simplex tableau as soon as they have been eliminated from the basic feasible solution.

Tabular solution for Example 4/8

Old S_1 row	–	key #	x	new S_4 values	=	new S_1 row
75	–	3	x	25/2	=	75/2
0	–	3	x	0	=	0
2	–	3	x	-3/8	=	25/8
1	–	3	x	0	=	1
0	–	3	x	0	=	0
0	–	3	x	1/8	=	-3/8
3	–	3	x	1	=	0
Old S_2 row	**–**	**key #**	**x**	**new S_4 values**	**=**	**new S_2 row**
20	–	0	x	25/2	=	20
0	–	0	x	0	=	0
1	–	0	x	-3/8	=	1
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	1/8	=	0
0	–	0	x	1	=	0
Old X_1 row	**–**	**key #**	**x**	**new S_4 values**	**=**	**new X_1 row**
25	–	-1	x	25/2	=	75/2
1	–	-1	x	0	=	1
1	–	-1	x	-3/8	=	5/8
0	–	-1	x	0	=	0
0	–	-1	x	0	=	0
0	–	-1	x	1/8	=	1/8
-1	–	-1	x	1	=	0

Tabular solution for Example 4/9

3rd Tableau

C_j			50	40	0	0	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	bi / a_{ij}
0	S_1	75/2	0	25/8	1	0	-3/8	0	12 Min. leaving
0	S_2	20	0	1	0	1	0	0	20
0	S_4	25/2	0	-3/8	0	0	1/8	1	---
50	X_1	75/2	1	5/8	0	0	1/8	0	60
	Z_j	1875	50	250/8	0	0	50/8	0	
	$C_j - Z_j$		0	70/8	0	0	-50/8	0	

Max. (Entering)

One more iteration is required. This time X2 comes into the solution and S1 is eliminated. After performing this iteration, the following simplex tableau shows that the optimal solution has been reached.

Tabular solution for Example 4/10

C_j			50	40	0	0	0	0
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4
40	X_2	12	0	1	8/25	0	-3/25	0
0	S_2	8	0	0	-8/25	1	3/25	0
0	S_4	17	0	0	3/25	0	2/25	1
50	X_1	30	1	0	-5/25	0	5/25	0
	Z_j	1980	50	40	14/5	0	26/5	0
	$C_j - Z_j$		0	0	-14/5	0	-26/5	0

It turns out that the optimal solution has been reached. (All $C_j - Z_j \leq 0$ and all artificial variables have been eliminated.)

EQUALITY CONTRAINTS_NEGATIVE RIGHT-HAND SIDE VALUES

- Simply add an artificial variable A1 to create a basic feasible solution in the initial simplex tableau.

  $6X1 + 4X2 - 5X3 = 30 \Rightarrow 6X1 + 4X2 - 5X3 + 1A1 = 30$

- One of the properties of the tableau form of a linear program is that the values on the right-hand sides of the constraints have to be nonnegative.
- e.g. # of units of the portable model (X2) has to be less than or equal to the # of units of the deskpro model (X1) after setting aside 5 units of the deskpro for internal company use.

  $X2 \leq X1 - 5$

- $- X1 + X2 \leq -5$
- (Min)Multiply by –1 $\Rightarrow$ (Max) $X1 - X2 \geq 5$
- We now have an acceptable nonnegative right-hand-side value. Tableau form for this constraint can now be obtained by subtracting a surplus variable and adding an artificial variable.

Tabular solution for Example 5/1

- Livestock Nutrition Co. produces specially blended feed supplements. LNC currently has an order for 200 kgs of its mixture.
- This consists of two ingredients

  X_1 (a protein source)

  X_2 (a carbohydrate source)
- The first ingredient, X_1 costs LNC 3MU a kg. The second ingredient, X_2 costs LNC 8MU a kg. The mixture can't be more than 40% X_1 and it must be at least 30% X_2.
- LNC's problem is to determine how much of each ingredient to use to minimize cost.

Tabular solution for Example 5/2

- The cost function can be written as

  Cost = $3X_1 + 8X_2$ Min!

- LNC must produce 200 kgs of the mixture – no more, no less.

  $X_1 + X_2 = 200$ kgs

- The mixture can't be more than 40% X_1, so we may use less than 80 kgs. (40% X 200 = 80). However, we must not exceed 80 kgs.

  $X_1 \leq 80$ kgs

- The mixture must be at least 30% X_2. Thus we may use more than 60 kgs, not less than 60 kgs. (30% X 200 = 60)
- $X_2 \geq 60$ kgs

Tabular solution for Example 5/3

Minimize : Cost = 3MU X1 + 8MU X2

Subject to X1 + X2 = 200 kgs

$X1 \leq 80$ kgs

$X2 \geq 60$ kgs

$X1 , X2 \geq 0$

- An initial solution: X1 + X2 = 200 kgs

$$\Rightarrow \quad X1 + X2 + A1 = 200$$

$$\Downarrow$$

- *Artificial variable* : A very expensive substance must not be represented in optimal solution.

Tabular solution for Example 5/4

- ***An artificial Variable*** is only of value as a computational device; it allows 2 types of restrictions to be treated.
  - The equality type
  - $\geq$ type
  - $X_1 \leq$ 80 kgs constraint on protein

    $\Rightarrow \quad X_1 + S_1 = 80$ kgs

$X_2 \geq$ 60 kgs constraint on carbohydrates

$\Rightarrow$ X2 - S2 + A2 = 60

- X1 , X2 , S1, S2, A1, A2 $\geq$ 0

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

0MU 0MU M M

Tabular solution for Example 5/5

Minimize : Cost $= 3X1 + 8X2 + 0S1 + 0S2 + MA1 + MA2$

Subject to : $X1 + X2 + A1 = 200$

$X1 + S1 = 80$

$X2 - S2 + A2 = 60$

All variables ≥ 0

Tabular solution for Example 5/6

Initial Tableau

C_j			3 MU	8 MU	M MU	0 MU	0 MU	M MU	
	Product mix	Quantity b_i	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	200	1	1	1	0	0	0	200/1=200
0	S_1	80	1	0	0	1	0	0	--
M	A_2	60	0	1	0	0	-1	1	60/1=60 Min. replaced row
	Z_j	260M	M	2M	M	0	-M	M	
	$C_j - Z_j$		3-M	8-2M	0	0	M	0	

Optimal column

Tabular solution for Example 5/7

Computation for 2nd tableau:

Replacing row = new X_2 values : 60/1=60, 0/1=0, 1/1=1, 0/1=0, -1/1=-1, 1/1=1

Old A_1 row	−	key #	x	new X_2 values	=	new A_1 row
200	−	1	x	60	=	140
1	−	1	x	0	=	1
1	−	1	x	1	=	0
1	−	1	x	0	=	1
0	−	1	x	0	=	0
0	−	1	x	-1	=	1
0	−	1	x	1	=	-1

Old S_1 row	−	key #	x	new X_2values	=	new S_1 row
80	−	0	x	60	=	80
1	−	0	x	0	=	1
0	−	0	x	1	=	0
0	−	0	x	0	=	0
1	−	0	x	0	=	1
0	−	0	x	-1	=	0
0	−	0	x	1	=	0

Tabular solution for Example 5/8

2nd Tableau

C_j			3	8	M	0	0	M	
	Product mix	Quantity b_i	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	140	1	0	1	0	1	-1	140/1=140
0	S_1	80	1	0	0	1	0	0	80/1=80 replaced row, min.
8	X_2	60	0	1	0	0	-1	1	60/0= --
	Z_j	140M + 480	M	8	M	0	M-8	8-M	
	$C_j - Z_j$		3-M	0	0	0	8-M	2M-8	

Optimal column

Tabular solution for Example 5/9

Computations for 3rd Tableau

Replacing row = new X_1 values: 80/1=80, 1/1=1, 0/1=0, 0/1=0, 1/1=1, 0/1=0, 0/1=0

Old A_1 row	–	key #	x	new X_1 values	=	new A_1 row
140	–	1	x	80	=	60
1	–	1	x	1	=	0
0	–	1	x	0	=	0
1	–	1	x	0	=	1
0	–	1	x	1	=	-1
1	–	1	x	0	=	1
-1	–	1	x	0	=	-1

Old X_2 row	–	key #	x	new X_1 values	=	new X_2 row
60	–	0	x	80	=	60
0	–	0	x	1	=	0
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	1	=	0
-1	–	0	x	0	=	-1
1	–	0	x	0	=	1

Tabular solution for Example 5/10

3rd Tableau

C_j			3	8	M	0	0	M	
	Product mix	Quantity bi	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	60	0	0	1	-1	1	-1	60/1=60 replaced row
3	X_1	80	1	0	0	1	0	0	80/0= --
8	X_2	60	0	1	0	0	-1	1	60/-1= - 60 not considered
	Z_j	60M – 720	3	8	M	3-M	M-8	8-M	
	$C_j – Z_j$		0	0	0	M-3	8-M	2M-8	

Optimal column

Tabular solution for Example 5/11

Computations for the 4th Tableau

Replacing row = new S_2 values: 60/1=60, 0/1=0, 0/1=0, 1/1=1, -1/1=-1, 1/1=1, -1/1=-1

Old X_1 row	–	key #	x	new S_2 values	=	new X_1 row
80	–	0	x	60	=	80
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	1	=	0
1	–	0	x	-1	=	1
0	–	0	x	1	=	0
0	–	0	x	-1	=	0

Old X_2 row	–	key #	x	new S_2 values	=	new X_2 row
60	–	-1	x	60	=	120
0	–	-1	x	0	=	0
1	–	-1	x	0	=	1
0	–	-1	x	1	=	1
0	–	-1	x	-1	=	-1
-1	–	-1	x	1	=	0
1	–	-1	x	-1	=	0

Tabular solution for Example 5/12

4th Tableau

C_j			3	8	M	0	0	M
	Product mix	Quantity bi	X_1	X_2	A_1	S_1	S_2	A_2
0	S_2	60	0	0	1	-1	1	-1
3	X_1	80	1	0	0	1	0	0
8	X_2	120	0	1	1	-1	0	0
	Z_j	1200	3	8	8	-5	0	0
	$C_j - Z_j$		0	0	M-8	5	0	M

∵ No negative values remain in the $C_j - Z_j$ row, we have reached the OPTIMAL solution.

It is to use 80 kgs of X_1 and 120 kgs of X_2. This results in a cost of 1200MU.
S_2 represents the amount of X_2 used over the minimum quantity required (60kg)

$$X_2 - S_2 + A_2 = 60 \longrightarrow 120 - 60 + 0 = 60 \longrightarrow 60 = 60 \because A_2 = 0.$$